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TWO PROOFS OF THE $P = W$ CONJECTURE
[*after* Maulik–Shen and Hausel–Mellit–Minets–Schiffmann]

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Introduction

For a smooth projective complex algebraic curve C, non-abelian Hodge theory of Donaldson (1983, 1987) and Hitchin (1987b), Corlette (1988) and Simpson (1988, 1992, 1994a) provides a real analytic isomorphism

$$\mathscr{M}^{\mathrm{Dol}}_{n,d}(C) \simeq \mathscr{M}^{\mathrm{B}}_{n,d}(C) \tag{1}$$

where $\mathscr{M}^{\mathrm{Dol}}_{n,d}(C)$ is the *Dolbeault* moduli space of rank n degree d semistable *Higgs bundles* $(E, \theta\colon E \to E \otimes \omega_C)$ on C and $\mathscr{M}^{\mathrm{B}}_{n,d}(C)$ is the *Betti* moduli space of d-twisted n-dimensional representations of the fundamental group $\pi_1(C)$.

These moduli spaces are complex algebraic varieties, and are smooth when n and d are coprime; however, the above isomorphism is transcendental and does not preserve the complex structures on these moduli spaces. In particular, the Dolbeault moduli space depends on the complex structure of C, whereas the Betti moduli space only depends on the topology (i.e. the genus) of C. The Betti moduli space is affine via its construction as an affine geometric invariant theory quotient, while the Dolbeault moduli space contains many compact subvarieties that appear as fibres of the *Hitchin map*. The Hitchin map is a proper morphism $h\colon \mathscr{M}^{\mathrm{Dol}}_{n,d}(C) \to \mathscr{A}_n$ to an affine space which records the coefficients of the characteristic polynomial of the Higgs field $\theta\colon E \to E \otimes \omega_C$ viewed as a twisted endomorphism of the vector bundle E. By the spectral correspondence (Beauville, Narasimhan, and Ramanan, 1989; Hitchin, 1987b; Schaub, 1998), the general fibres $h^{-1}(a)$ are abelian varieties isomorphic to the Jacobian of an associated spectral curve $\mathscr{C}_a \subset T^*C := \mathrm{Tot}(\omega_C)$, which is a degree n cover of C.

The homeomorphism (1) induces an isomorphism on cohomology, but this does not preserve mixed Hodge structures. Assume from now on that n and d are coprime. Then the Hodge structure of the Dolbeault moduli space is pure, as there is a $\mathbb{G}_m$-action which scales the Higgs field and provides a deformation retract onto its fixed locus, which is a smooth projective subvariety. The mixed Hodge structure on the Betti moduli space is genuinely mixed and does not see the complex structure of C; in fact, it turns out to be of Hodge–Tate type. Consequently, one can ask if the non-trivial weight filtration on the cohomology of the Betti moduli space has a geometric interpretation on the Dolbeault side via non-abelian Hodge theory.

Hausel and Rodriguez-Villegas (2008) spotted a symmetry in the Hodge numbers of Betti moduli spaces and formulated a Curious Hard Lefschetz conjecture relating opposite graded pieces of the weight filtration; they proved this in rank 2 and Mellit (2019) recently proved this in higher ranks. This lead de Cataldo, Hausel, and Migliorini (2012) to formulate (and prove in rank $n = 2$) the $P = W$ *conjecture*: the *weight filtration* W on the cohomology of the Betti moduli space should correspond to the *perverse Leray filtration* P associated to the Hitchin map on the Dolbeault moduli space. This would imply that the Curious Hard Lefschetz property is explained by the Relative Hard Lefschetz Theorem for the Hitchin map. In 2022, two independent proofs of the $P = W$ conjecture were given by Maulik–Shen (MS) and Hausel–Mellit–Minets–Schiffmann (HMMS).

Although these two proofs are very different, they share a common starting point: the cohomology of the Dolbeault moduli space is generated by certain *tautological classes* obtained from Künneth components of the Chern classes of the universal Higgs bundle $\mathscr{E} \to \mathscr{M}_{n,d}^{\mathrm{Dol}} \times C$ by work of Markman (2002). If these tautological classes are appropriately normalised, their weights are known by work of Shende (2017). Since the weight filtration is multiplicative (i.e. the cup product adds weights), the $P = W$ conjecture can be studied entirely on the Higgs moduli space as a question about the interaction between products of tautological classes and the perverse filtration. One key challenge is that the perverse filtration is not in general multiplicative.

Maulik and Shen reduce the $P = W$ conjecture to a sheaf-theoretic property they call *strong perversity* for cup products with tautological classes in the derived category of constructible sheaves over the Hitchin base. To prove that the Chern classes of the universal bundle have the predicted strong perversity, they work with parabolic moduli spaces appearing in Yun's *global Springer theory* (Yun, 2011, 2012). The parabolic structure is given by a full flag in a fibre and induces certain tautological line bundles which have the desired strong perversity over an open in the Hitchin base by work of Yun. Using the decomposition theorem (Beĭlinson, Bernstein, and Deligne, 1982), this would automatically extend to the whole Hitchin base if the parabolic Hitchin map had full supports. Unfortunately, this is not the case, but previous work of Maulik and Shen (2021) that builds on work of Ngô (2006, 2010) and Chaudouard and Laumon (2016) offers a solution: instead of classical Higgs bundles, consider D-twisted Higgs bundles $(E, \theta \colon E \to E \otimes \omega_C(D))$ for an effective divisor D. Maulik and Shen prove a parabolic support theorem for D of sufficiently high degree and then pass from D-twisted Higgs bundles to classical Higgs bundles via a *vanishing cycles* argument as in Maulik and Shen (2021). Consequently, they deduce the image of the weight filtration under non-abelian Hodge theory is contained in the perverse filtration and conclude they are equal using the Relative and Curious Hard Lefschetz Theorems.

Hausel, Mellit, Minets and Schiffmann approach the $P = W$ conjecture by constructing Lie algebras acting on the cohomology using two natural operations on the cohomology of Higgs moduli spaces (or rather their stacks): cup products with tautological classes and *Hecke correspondences* that modify a vector bundle at single point in C. By the Relative Hard Lefschetz Theorem for the Hitchin fibration, a choice of relatively ample divisor class provides an identification between opposite graded pieces of the perverse filtration and defines a nilpotent operator on the associated graded vector space that can be extended to an $\mathfrak{sl}_2$-triple $\langle \mathbf{e}, \mathbf{f}, \mathbf{h} \rangle$ whose $\mathbf{h}$-graded pieces are the perverse graded pieces. The idea is to find a lifted $\mathfrak{sl}_2$-triple acting on the cohomology (rather than the associated graded object for P) where the first two operators come from tautological and Hecke operators respectively, the third induces the perverse filtration and the tautological classes are $\mathbf{h}$-eigenvectors in order to describe the interaction between tautological classes and the perverse filtration. The desired $\mathfrak{sl}_2$-triple is found by considering an action of a much larger Lie algebra $\mathscr{H}_2$ of polynomial Hamiltonian vector fields on the plane and by using the spectral correspondence to instead work with sheaves on surfaces and their cohomological Hall algebras as in Mellit et al. (2023). In this story, there is again a technical problem to overcome: Hecke correspondences do not preserve semistability and so first a result is shown on the *elliptic* locus, where the spectral curves are integral, and then parabolic bundles are used to pass from the elliptic locus to the whole moduli space. This proof actually shows the perverse and weight filtrations both coincide with the filtration induced by this $\mathfrak{sl}_2$-triple and also gives a new proof of the Curious Hard Lefschetz Theorem.

Recently, a third proof due to Maulik, Shen, and Yin (2023) appeared, which offers a new perspective; we do not include the details due to lack of time and space but merely mention the main results. A key theorem is that the elliptic locus is a *twisted (self)-dualisable abelian fibration satisfying Fourier vanishing*; this builds on Arinkin's work on compactified Jacobians of locally planar integral curves (Arinkin, 2013). For any such fibration, Maulik, Shen, and Yin (2023) prove a (motivic) decomposition and show that the perverse filtration is multiplicative; hence the tautological operators have the expected perversity and the *Chern filtration* (see §2.4.1) is contained in the perverse filtration, which suffices to conclude the equality $P = W$ using the Curious Hard Lefschetz Theorem.

Although the methods of proofs are very different, they involve similar moduli spaces (for example, parabolic Higgs bundles and elliptic loci appear in all three proofs). The third proof (Maulik, Shen, and Yin, 2023) may come closer to relating to the approach in (HMMS) as follows. As a consequence of the semi-classical limit of the geometric Langlands correspondence, the Hitchin fibration is expected to be self-dual, in the sense that there is a derived equivalence which should exchange Hecke operators and certain tautological operators. This expectation is realised over the regular locus by Donagi and Pantev (2012) and over the elliptic locus by Arinkin (2013). Furthermore, Polishchuk (2007) produced actions of $\mathfrak{sl}_2$ (and $\mathscr{H}_2$) on the rational tautological Chow ring of the Jacobian of a curve using the Fourier transform, which suggests a possible deeper relation between Maulik, Shen, and Yin (2023) and (HMMS).

The goal of this report is to give an overview of the main ideas in the proofs of Maulik–Shen (MS) and Hausel–Mellit–Minets–Schiffmann (HMMS), and the relevant background.

Many of the techniques and ideas in the proofs play an important role in the study of moduli spaces more generally: operators constructed from tautological classes and natural correspondences generate actions of interesting algebras in various contexts, such as Hilbert schemes (Grojnowski, 1996; Nakajima, 1997), infinite symmetric powers (Kimura and Vistoli, 1996), Jacobians (Polishchuk, 2007) and CoHAs (Mellit et al., 2023). Furthermore vanishing cycles, support theorems and duality for abelian varieties are important tools for studying (Higgs) moduli spaces.

Structure of the paper. — In §1, we introduce the moduli spaces and recall non-abelian Hodge theory, then provide some background on mixed Hodge structures and perverse sheaves in order to state the $P = W$ conjecture. In §2, we introduce the tautological classes and state Markman's result on tautological generation and Shende's computation of the weights of the tautological generators; this leads to various reformulations of the $P = W$ conjecture purely in terms of the interaction of the perverse filtration with tautological classes on the Dolbeault side. Finally in §3 and in §4 respectively, we discuss the proofs of Maulik–Shen (MS) and Hausel–Mellit–Minets–Schiffmann (HMMS).

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1. Moduli spaces and the $P = W$ conjecture

Throughout, we let C denote a smooth projective geometrically connected algebraic curve of genus g over a field k, which will often be the complex numbers.

1.1. Moduli spaces and non-abelian Hodge theory

The terminology *non-abelian* Hodge theory is used to indicate a generalisation of classical Hodge theory, which relates various cohomology theories with coefficients in the abelian group $\mathbb{C}$, to a version taking values in the non-abelian coefficient group $\mathrm{GL}_n(\mathbb{C})$ and leads to diffeomorphism of various moduli spaces related to these cohomology theories.

For a smooth complex projective variety X, the de Rham isomorphism and Hodge decomposition give isomorphisms

$$H^k_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{C}) \simeq H^k_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X)$$

between the (topological) *Betti* cohomology of singular cochains on the underlying topological space X^{an}, the (differential geometric) *de Rham* cohomology groups and the (holomorphic) *Dolbeault* cohomology groups, which can be expressed as sheaf cohomology groups via the Dolbeault isomorphisms $H^{p,q}(X) \simeq H^q(X, \Omega^p_X)$. The Hodge decomposition involves relating both cohomology theories to harmonic operators; similarly, harmonic metrics also play a central role in the non-abelian version of this story.

Let us focus on the case of a smooth complex projective curve C. We have

$$H^1_{\mathrm{B}}(C^{\mathrm{an}}, \mathbb{C}) \simeq H^1_{\mathrm{dR}}(C^{\mathrm{an}}, \mathbb{C}) \simeq H^{0,1}(C) \oplus H^{1,0}(C) \simeq H^1(C, \mathscr{O}_C) \oplus H^0(C, \omega_C).$$

The idea of non-abelian Hodge theory is to replace the abelian coefficient group $\mathbb{C}$ with the non-abelian coefficient group $\mathrm{GL}_n(\mathbb{C})$. For $n = 1$, on the Betti side we have

$$H^1_B(C, \mathbb{C}^*) \simeq \mathrm{Hom}(\pi_1(C), \mathbb{C}^*) = \mathrm{Rep}(\pi_1(C), \mathbb{C}^*)$$

characters of the fundamental group and on the Dolbeault side we have pairs consisting of a holomorphic line bundle parametrised by $H^1(C, \mathscr{O}^*_C) \simeq \mathrm{Pic}(C)$ and a holomorphic 1-form parametrised by $H^0(C, \omega_C)$. For $n > 1$, the Betti moduli space $\mathscr{M}^{\mathrm{B}}_{\mathrm{GL}_n}(C)$ parametrises representations $\pi_1(C) \to \mathrm{GL}_n(\mathbb{C})$ and the Dolbeault moduli space $\mathscr{M}^{\mathrm{Dol}}_{n,0}(C)$ parametrises certain rank n degree 0 holomorphic vector bundles on C equipped with a Higgs field, which is a bundle endomorphism twisted by the canonical line bundle ω_C. We formally introduce these moduli spaces below. The non-abelian Hodge theorem gives a real analytic isomorphism between the Dolbeault and Betti moduli spaces, which is not compatible with their complex structures. The proof uses the existence of certain harmonic metrics and a third de Rham moduli space of holomorphic bundles with connections appears; however, we will not introduce the