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ORIENTABILITY OF THE MODULI SPACE OF REAL MAPS AND REAL GROMOV–WITTEN THEORY [*after* Penka Georgieva and Aleksey Zinger]

by **Michele Ancona**

I knew exactly what to do,
but in a much more real sense
I had no idea of what to do.
— Michael Scott

Introduction

Gromov–Witten theory studies symplectic manifolds via maps from Riemann surfaces into them. Counting such maps in a proper way produces rational numbers, called Gromov–Witten (GW-) invariants, that are invariant by deformation of a symplectic manifold. Let us give an informal definition of GW-invariants of a symplectic manifold (X, ω). One first fixes a generic almost-complex structure J on (X, ω) such that $\omega(\cdot, J\cdot)$ defines a Riemannian metric. Such an almost-complex structure is called calibrated. Gromov (1985) proved that the space of calibrated almost-complex structures is non-empty and contractible. Then, for any non-negative integers g and k and any homology class $A \in H_2(X, \mathbb{Z})$, one considers the moduli space $\overline{\mathscr{M}}_{g,k}(X, A)$ consisting of elements $[u, (\Sigma, j), x_1, \ldots, x_k]$ where:

- ▷ Σ is a genus g compact surface with at worst nodal singularities and j is a complex structure on Σ (the pair (Σ, j) is called a *Riemann surface*);
- ▷ $x_1, \ldots, x_k$ are marked points on Σ;
- ▷ $u \colon (\Sigma, j) \to (X, J)$ is a map verifying $J \circ \mathrm{d}u = \mathrm{d}u \circ j$ (such a map is called *J-holomorphic* or *pseudo-holomorphic*) and such that the push-forward $u_*[\Sigma]$ of the fundamental class of Σ is A (one says that *u realizes A*);
- ▷ the group of automorphisms of $(u, (\Sigma, j), x_1, \ldots, x_k)$ (that is, the biholomorphisms φ of (Σ, j) such that $\varphi(x_i) = x_i$ and $u \circ \varphi = u$) is finite.

One then fixes cohomology classes $\alpha_1, \dots, \alpha_k$ of X such that

$$\sum_{i=1}^{k} \deg(\alpha_i) = \dim \overline{\mathscr{M}}_{g,k}(X, A) = (1-g)(6 - \dim X) + 2\langle c_1(TX), A\rangle + 2k$$

and counts the number of maps $[u, (\Sigma, j), x_1, \dots, x_k] \in \overline{\mathscr{M}}_{g,k}(X, A)$ such that $u(x_i) \in Y_i$, where $Y_i \subset X$ is a generic representative of the Poincaré dual of α_i. The number of such maps is then independent of the choice of J and of the representatives of the Poincaré duals of the classes α_i and is called the GW-invariant $GW_{g,A}(\alpha_1, \dots, \alpha_k)$. For example, the number N_d of degree d rational curves in $\mathbb{P}^2$ passing through a collection of $3d-1$ generic points is a GW-invariant of $\mathbb{P}^2$, namely $N_d = GW_{0,d}(\mathrm{pt}, \dots, \mathrm{pt})$. The equalities $N_1 = 1$ and $N_2 = 1$ are evident and $N_3 = 12$ can be proved by counting the number of singular fibers of the pencil of cubics passing through 8 generic points of $\mathbb{P}^2$. The number $N_4 = 620$ was obtained by Zeuthen (1873). We had to wait until the mid '90s to obtain the value of N_d for any d. This was a consequence of the work of Kontsevich who found the beautiful recursive formula

$$N_d = \sum_{\substack{d_A + d_B = d \\ d_A, d_B \geqslant 1}} N_{d_A} N_{d_B} \left(d_A^2 d_B^2 \binom{3d-4}{3d_A - 2} - d_A^3 d_B \binom{3d-4}{3d_A - 1} \right)$$

which allows us to compute N_d for any d from the value $N_1 = 1$ (see Kontsevich and Manin, 1994). Such a formula was indeed found thanks to the interpretation of the numbers N_d as Gromov–Witten invariants and it actually expresses the associativity of the product in the quantum cohomology ring of $\mathbb{P}^2$.

Remark. Witten (1991) discovered that the coefficients of the quantum multiplication in quantum cohomology could be defined mathematically using symplectic geometry, in particular using intersection theory on the space of holomorphic curves in an algebraic or symplectic manifold. It was Gromov (1985), some years before, who introduced the notion of pseudo-holomorphic curves in symplectic geometry. For these reasons the invariants we are talking about are called Gromov–Witten invariants. The first mathematical foundations of Gromov–Witten theory are the works of Kontsevich and Manin (1994) in the algebraic setting and of Ruan and Tian (1995) in the symplectic one.

A *real symplectic manifold* is a triple (X, ω, σ_X) where (X, ω) is a symplectic manifold and $\sigma_X \colon X \to X$ is an involution verifying $\sigma_X^* \omega = -\omega$, called the *real structure*. We will always assume that X is compact. The main example is the complex projective space $\mathbb{P}^n$ equipped with the Fubini–Study form ω_{FS} and with the standard conjugation $\mathrm{conj} : \mathbb{P}^n \to \mathbb{P}^n$ sending $[z_0 : \cdots : z_n]$ to $[\bar{z}_0 : \cdots : \bar{z}_n]$. More generally, if a projective manifolds $X \subset \mathbb{P}^n$ is defined by real polynomial equations, then $(X, \omega_{\mathrm{FS}|X}, \mathrm{conj}_{|X})$ is a real symplectic manifold. The *real locus* of a real symplectic manifold is by definition the fixed locus of σ_X and is denoted by $\mathbb{R}X$. It is either empty or a finite union of Lagrangian submanifolds of (X, ω). A *real Riemann surface* (Σ, σ, j) is a Riemann surface (Σ, j) equipped with an anti-holomorphic involution σ. Given a calibrated almost-complex structure J on (X, ω) verifying $\sigma_X^* J = -J$, a *real curve in* (X, ω, σ_X) is a (σ, σ_X)-equivariant J-holomorphic map from a real Riemann surface (Σ, σ, j) into (X, σ_X, J). As for the complex case, one would like to extract invariants of (X, ω, σ_X) from counting real curves inside it. However, the number of real curves realizing a given class and passing through an appropriate number of cycles $Y_i \subset X$ depends on the particular choice of the cycles, and not just on their (co)-homology classes. For example, the number of degree d real rational curves $u\colon (\mathbb{P}^1, \mathrm{conj}) \to (\mathbb{P}^2, \mathrm{conj})$ passing through $3d-1$ generic points of $\mathbb{R}\mathbb{P}^2$ depends on the choice of such points. The first breakthrough was made by Welschinger (2005a,b, 2007a) when he defined invariants of real symplectic fourfolds and strongly semipositive sixfolds, now called Welschinger invariants. The approach of Welschinger was to assign a sign ± 1 to each individual real rational curve passing through a fixed real configuration of points (i.e. a collection of r real points on a connected component $\mathbb{R}X_0$ of $\mathbb{R}X$ and l pairs of complex-conjugate points in X) and by proving that the resulting *signed* count of such curves is invariant, that is, it does not depend on the position of the points but only on the chosen connected component $\mathbb{R}X_0$ of $\mathbb{R}X$, on r and on l. By their own definition, Welschinger invariants give lower bounds for the number of real rational curves passing through a generic real configuration of points. We will not define Welschinger invariants here, but refer the reader to the Bourbaki seminar of Oancea (2012) for a gentle introduction to them. Since the discovery of Welschinger invariants, many advances have been made on real Gromov–Witten theory in genus 0, but essentially none in higher genus.

Remark. The Welschinger sign of a real curve inside a real symplectic manifold makes sense for real curves of any genus; however the resulting signed count is *not* invariant in higher genus (see for example Welschinger (2005a) and Itenberg, Kharlamov, and Shustin (2003, Theorem 3.1)).

Let us explain one of the main difficulties that occurs in trying to define real Gromov–Witten invariants in general. For this, let us notice that the (complex) GW-invariant $GW_{g,A}(\alpha_1, \ldots, \alpha_k)$ described above coincides with the integral

$$\int_{\overline{\mathscr{M}}_{g,k}(X,A)} \mathrm{ev}_1^* \alpha_1 \wedge \cdots \wedge \mathrm{ev}_k^* \alpha_k$$

where $\mathrm{ev}_i\colon [u, (\Sigma, j), x_1, \ldots, x_k] \in \overline{\mathscr{M}}_{g,k}(X, A) \mapsto u(x_i) \in X$. For the integral to be well-defined, one needs the space $\overline{\mathscr{M}}_{g,k}(X, A)$ to be oriented. Here is one of the main problems in real Gromov–Witten theory: the moduli spaces of real J-holomorphic curves in (X, ω, σ_X) are in general not orientable, and when they are, there is not a preferred orientation. The orientability problem is then a central question in real

Gromov–Witten theory. Welschinger invariants have been interpreted and studied in term of orientability of moduli spaces of pseudo-holomorphic disks by Cho (2008) and Solomon (2006) using the work of Fukaya et al. (2009), in particular the notion of relative spin structure (we will recall this notion later in the introduction). Solomon extended the definition of these invariants to real symplectic sixfolds and for real curves of higher genus but with fixed conformal structure. Later, Georgieva (2016) defined a signed count of real genus 0 curves with conjugate pairs of arbitrary constraints in arbitrary dimensions for strongly semipositive manifolds (X, ω) verifying some additional topological properties which, in particular, implies the existence of a relative spin structure on $\mathbb{R}X$. Such invariants were further generalized by Farajzadeh Tehrani (2016) who included also genus 0 real curves with empty real locus in the signed count.

The main theorem we present in this note is a theorem by Georgieva and Zinger (2018), which gives sufficient conditions on a real symplectic manifold (X, ω, σ_X) for the moduli spaces $\mathbb{R}\overline{\mathscr{M}}_{g,l}(X, A)$ of real maps from genus g real curves together with l pairs of complex-conjugate marked points to be oriented for any g, l and class $A \in H_2(X, \mathbb{Z})$. The sufficient condition is given by the notion of real-orientation on (X, ω, σ_X) defined below in the introduction. The main theorem (Theorem 3.4) then asserts that a real-orientation on a real-orientable symplectic manifold (X, ω, σ_X) of dimension $2n$, with $n \notin 2\mathbb{N}$, orients $\mathbb{R}\overline{\mathscr{M}}_{g,l}(X, A)$. An orientation of $\mathbb{R}\overline{\mathscr{M}}_{g,l}(X, A)$ can then be used to define genus g real Gromov–Witten invariants of (X, ω, σ_X) with conjugate pairs of contraints.

In order to introduce the notion of real orientability, we first need the following definition.

Definition. *A real bundle pair (E, σ_E) over (X, σ_X) is a complex vector bundle $\pi\colon E \to X$ equipped with an involution σ_E which is complex anti-linear in the fibers and such that $\pi \circ \sigma_E = \sigma_X \circ \pi$. Such involution is called a real structure of E.*

An isomorphism of real bundle pairs is an isomorphism between the underlying complex vector bundles which commutes with the real structures.

The fixed locus $\mathbb{R}E$ of (E, σ_E) is then a real vector bundle over $\mathbb{R}X$ whose real rank equals the complex rank of E. For example, the tangent bundle $(TX, d\sigma_X)$ of (X, σ_X) is a real bundle pair over (X, σ_X). Tensor products, direct sums, duals and exterior powers of real bundle pairs are again real bundle pairs.

Definition (Real orientability). *A real symplectic manifold is real-orientable if there exists a rank 1 real bundle pair (L, σ_L) over (X, σ_X) such that*

(1) $w_2(T\mathbb{R}X) = w_1(\mathbb{R}L)^2$, *where $w_i(\cdot) \in H^i(\mathbb{R}X, \mathbb{Z}/2)$ denotes the i-th Stiefel–Whitney class of a real vector bundle;*

(2) $\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX, d\sigma_X)$ *is isomorphic (as a real bundle pair) to $(L, \sigma_L)^{\otimes 2}$.*

Here are some examples of real-orientable symplectic manifolds:

- ▷ The odd-dimensional projective space $(\mathbb{P}^{2n-1}, \omega_{\mathrm{FS}}, \mathrm{conj})$. In this case, one has $\Lambda^{\mathrm{top}}_{\mathbb{C}}(T\mathbb{P}^{2n-1}, \mathrm{conj}) = (\mathscr{O}_{\mathbb{P}^{2n-1}}(2n), \sigma_{2n})$ and $(L, \sigma_L) = (\mathscr{O}_{\mathbb{P}^{2n-1}}(n), \sigma_n)$, where σ_k is the natural real structure of $\mathscr{O}_{\mathbb{P}^{2n-1}}(k)$ over $(\mathbb{P}^{2n-1}, \mathrm{conj})$.
- ▷ The projective space $(\mathbb{P}^{4n-1}, \omega_{\mathrm{FS}}, \tau)$ with empty real locus. Here, τ maps a point $[x_0 : x_1 : \cdots : x_{4n-2} : x_{4n-1}]$ to $[\bar{x}_1 : -\bar{x}_0 \cdots : \bar{x}_{4n-1} : -\bar{x}_{4n-2}]$. In this case, we have $\Lambda^{\mathrm{top}}_{\mathbb{C}}(T\mathbb{P}^{4n-1}, \mathrm{d}\tau) = (\mathscr{O}_{\mathbb{P}^{4n-1}}(4n), \tau_{4n})$ and $(L, \sigma_L) = (\mathscr{O}_{\mathbb{P}^{4n-1}}(2n), \tau_{2n})$, where τ_{2k} is the natural real structure of $\mathscr{O}_{\mathbb{P}^{4n-1}}(2k)$ over $(\mathbb{P}^{4n-1}, \tau)$. Remark that the line bundle $\mathscr{O}_{\mathbb{P}^{4n-1}}(2k+1)$ over $(\mathbb{P}^{4n-1}, \tau)$ does not admit any real structure.
- ▷ Complete intersections $X \subset \mathbb{P}^n$ defined by $n-3$ real polynomials of degrees $d_1, \ldots, d_{n-3}$ with $d_1 + \cdots + d_{n-3} \equiv n+1 \pmod 4$. Indeed, the adjunction formula says that $\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ is isomorphic to $(\mathscr{O}_X(n_d), \sigma_{n_d})$, with $n_d := n + 1 - d_1 - \cdots - d_{n-3}$ and, under the previous assumption, the real bundle pair $(L, \sigma_L) = (\mathscr{O}_X(n_d/2), \sigma_{n_d/2})$ verifies the two real orientability conditions. An example of such real symplectic manifold is a real quintic threefold in $\mathbb{P}^4$.
- ▷ Real compact Kähler Calabi–Yau threefolds and, more generally, real compact Kähler Calabi–Yau manifolds with spin real locus. In this case, $\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ is trivial so that the real bundle pair $(L, \sigma_L) = \Lambda^{\mathrm{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ itself verifies the two real orientability conditions.

Remark. Recently, Georgieva and Ionel (2021) have defined the notion of twisted real-orientation, which is a slight generalization of the notion of real-orientation, and checked that the proofs of the main theorems of Georgieva and Zinger (2018, 2019a,b) can be adapted for twisted real-orientable symplectic manifolds of odd "complex" dimension. For example, all odd-dimensional projective spaces $(\mathbb{P}^{2n-1}, \omega_{\mathrm{FS}}, \tau)$ with empty real locus are twisted real-orientable, but they are not real-orientable. Very recently, Georgieva and Zinger (2023) gave more details about this and also corrected some minor errors in their previous articles.

Let us collect some remarks on the notion of real orientability. Let us start with the second point of the definition. An isomorphism between $\Lambda^{\mathrm{top}}_{\mathbb{C}}(TX, \mathrm{d}\sigma_X)$ and $(L, \sigma_L)^{\otimes 2}$ induces, by restriction to the real locus, an isomorphism of real line bundles over $\mathbb{R}X$ between $\Lambda^{\mathrm{top}}_{\mathbb{R}}(T\mathbb{R}X)$ and $(\mathbb{R}L)^{\otimes 2}$. Now, the line bundle $(\mathbb{R}L)^{\otimes 2}$ is orientable, so a necessary condition for a real symplectic manifold to be real-orientable is that its real locus is orientable. Let us now comment on the first point in the definition of real orientability. Recall that a real vector bundle V over a topological space M is orientable if and only if its first Stiefel–Whitney class $w_1(V) \in H^1(M, \mathbb{Z}/2)$ vanishes and that an orientable vector bundle V admits a spin structure if and only if