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EXOTIC SUBGROUPS OF HYPERBOLIC GROUPS

by **Olivier Guichard**

1. Introduction

Since their introduction by Gromov (1987), word hyperbolic groups have been the focus of a lot of activity and have proved central in attacking a number of problems. It was soon noticed that their cohomological properties are very strong. As a matter of fact, a torsion-free word hyperbolic group Γ is of type $\mathscr{F}$, meaning that Γ is the fundamental group of a finite aspherical cell complex (Gromov (1987) attributes this to Eliyahu Rips). Such a property for a group Γ is called a *finiteness* property. For every positive integer n, there is a coarser finiteness property denoted $\mathscr{F}_n$ that requires a group Γ to be the fundamental group of an aspherical cell complex, possibly infinite, but which has only finitely many cells up to dimension n. A group of type $\mathscr{F}_n$ and not of type $\mathscr{F}_{n+1}$ is sometimes said to have exotic finiteness properties(1). The aim of this report is to illustrate that word hyperbolic groups can have exotic subgroups: subgroups with exotic finiteness properties or subgroups of type $\mathscr{F}$ but not word hyperbolic.

Theorem 1.1 (Llosa Isenrich and Py, 2024, corollary 3). *Let n be a positive integer. There exists a word hyperbolic group Γ containing a subgroup that is of type $\mathscr{F}_n$ but not of type $\mathscr{F}_{n+1}$.*

Theorem 1.2 (Italiano, Martelli, and Migliorini, 2023, corollary 2). *There exists a word hyperbolic group Γ containing a subgroup of type $\mathscr{F}$ that is not word hyperbolic.*

In both statements the subgroups are kernels of homomorphisms from Γ to $\mathbf{Z}$ (and in particular are normal subgroups). The geometric counterparts of these homomorphisms are maps from M to the circle, where M is a manifold or a pseudo-manifold whose fundamental group is Γ. This makes the analysis of the subgroups amenable to Morse-theoretical techniques. More precisely, on one hand Lefschetz theory is used

(1)This terminology was coined down by Dimca, Papadima, and Suciu (2009, section 5), later used in a book review by Meier (2013), and popularized by Llosa Isenrich (2019); see also the title of section 7 in Jankiewicz, Norin, and Wise (2021). A formal definition appeared first in Llosa Isenrich and Py (2023).

by Llosa Isenrich and Py (2024) to study word hyperbolic groups which are arithmetic subgroups of $\mathrm{U}(n,1)$ (and M is the quotient of the unit ball in $\mathbf{C}^n$ by the action of these arithmetic subgroups); on the other hand the Morse theory of affine cell complexes developed by Bestvina and Brady (1997) is used by Italiano, Martelli, and Migliorini (2023) for their word hyperbolic groups which are given by combinatorio-geometrical data.

As emphasized by the authors themselves (and apparent in that the above citations point to corollaries), the interesting statements may not be the above results, that give positive solutions to questions raised after the introduction of word hyperbolic groups, but the geometric constructions of which they are the shadows. The present report will indeed sketch these constructions and try to refer to the original articles for complete proofs.

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2. A brief overview of the historical development and further statements

Finiteness properties have various declinations: the properties $\mathrm{FH}_n(R)$ (R is a ring) request that the group Γ is the fundamental group of a compact cell complex whose universal cover has trivial reduced homology with coefficients in R in degrees $< n$ (Bestvina and Brady, 1997, pp. 445–446), and the properties $\mathrm{FP}_n(R)$ request that the trivial $R[\Gamma]$-module has a projective resolution whose homogeneous factors of degree $\leqslant n$ are finitely generated. The property $\mathscr{F}_1$ is equivalent to the group Γ being finitely generated. The property $\mathscr{F}_2$ is equivalent to the group Γ being finitely presented. For all n, the property $\mathscr{F}_n$ implies the property $\mathrm{FH}_n(\mathbf{Z})$, $\mathrm{FH}_n(\mathbf{Z})$ implies $\mathrm{FH}_n(R)$, and $\mathrm{FH}_n(R)$ implies $\mathrm{FP}_n(R)$ (and $\mathrm{FP}_n(\mathbf{Z})$ implies $\mathrm{FP}_n(R)$).

Rips (1982, corollary (b)) constructed the first example of a finitely generated, hence $\mathscr{F}_1$, but not finitely presented, hence not $\mathscr{F}_2$, subgroup in a small cancellation group (in particular in a word hyperbolic group). In his essay, Gromov (1987, section 4.4.A) suggested a strategy for finding subgroups with exotic finiteness properties in a word hyperbolic group, by taking covers of a flat torus, ramified over a union of codimension 2 tori meeting orthogonally, and that fiber over the circle (later Mladen Bestvina showed that Gromov's construction does not lead to a word hyperbolic group; his argument is reproduced in Brady, Riley, and Short (2007, pp. 70–71)).

The question of the existence, in word hyperbolic groups, of subgroups of type $\mathscr{F}_n$ and not $\mathscr{F}_{n+1}$ was explicitely raised by Gersten (1995, p. 130) (who uses the notation FP_n instead of the now established notation $\mathscr{F}_n$). It was also stated by Brady (1999, question 7.1) who constructed finitely presented subgroups (hence of type $\mathscr{F}_2$) not of type $\mathscr{F}_3$[(2)]. More examples of finitely presented and not $\mathscr{F}_3$ subgroups, elaborating on Brady's construction and building on the Bestvina–Brady Morse theory (see below section 4.4), were subsequently obtained by Kropholler (2021), Kropholler and Llosa Isenrich (2023), and Lohda (2018). Llosa Isenrich, Martelli, and Py (2021) built the first example of a subgroup of type $\mathscr{F}_3$ and not $\mathscr{F}_4$ elaborating on a fibration of a complete, finite volume, hyperbolic 8-manifold constructed in Italiano, Martelli, and Migliorini (2022) and gave examples of subgroups of type $\mathrm{FP}_n(\mathbf{Q})$ and not $\mathrm{FP}_{n+1}(\mathbf{Q})$ in cubulable arithmetic lattices of the Lie group $\mathrm{O}(2n, 1)$.

Kernels of homomorphisms onto $\mathbf{Z}$ give examples of groups with intermediate finiteness properties. For example the kernel of the morphism from the free group $\mathbb{F}_2$ onto $\mathbf{Z}$ mapping all the generators to 1 is not finitely generated; the kernel of the morphism $\mathbb{F}_2 \times \mathbb{F}_2 \to \mathbf{Z}$ sending every generator to 1 is finitely generated but not finitely presented. Stallings (1963) gave the first example of a group of type $\mathscr{F}_2$ (thus finitely presented) that is not of type $\mathscr{F}_3$; it was later observed (Gersten, 1995) that this example is isomorphic to the kernel of the morphism from $(\mathbb{F}_2)^3$ to $\mathbf{Z}$ sending every generator to 1. For every positive integer n, the kernel of the similar homomorphism from $(\mathbb{F}_2)^n$ to $\mathbf{Z}$ is of type $\mathscr{F}_{n-1}$ and not of type $\mathscr{F}_n$ (Bieri, 1976).

On the other hand, the question (answered thus negatively by theorem 1.2) whether a subgroup of type $\mathscr{F}$ in a word hyperbolic group is itself hyperbolic can be traced back to Bestvina's problem list[(3)] and is also stated by Brady (1999, question 7.2). More recently the question appears in Jankiewicz, Norin, and Wise (2021, section 7). The techniques developed in this previous reference have been used by Italiano, Martelli, and Migliorini (2022; 2023) to construct fibrations of hyperbolic manifolds over the circle and the fibration of a pseudo-manifold explained below in section 4 that leads to theorem 1.2. Constructions of hyperbolic manifolds along the same line were also proposed in Kolpakov and Martelli (2013) and Kolpakov and Slavich (2016).

The related question whether a finitely presented subgroup of a word hyperbolic group of cohomological dimension 2 is itself hyperbolic has a positive answer (Gersten, 1996). The similar question in dimension 3 or 4 (is it true that an $\mathscr{F}_3$,

(2) Brady asks the existence of a finitely presented subgroup of type $\mathrm{FP}_n(\mathbf{Z})$ and not $\mathrm{FP}_{n+1}(\mathbf{Z})$. However for a finitely presented group, the implication $\mathrm{FP}_n(\mathbf{Z}) \Rightarrow \mathscr{F}_n$ holds (a proof can be found in the proof of theorem 7.1 in Brown (1982, chapter VIII), it relies on the Hurewicz theorem), thus Brady's question is indeed Gersten's question.

(3) Written in August 2000 and available at `https://www.math.utah.edu/~bestvina/`, retrieved on January 12th 2024.

resp. $\mathscr{F}_4$, subgroup of a hyperbolic group of cohomological dimension 3, resp. 4, is hyperbolic) is still open. In dimension 5, theorem 1.2 provides a counter-example.

The discussion so far emphasizes morphisms onto **Z**. Central objects which we will not discuss, but enable a finer understanding of the finiteness properties of the kernels of these morphisms, are the Bieri–Neumann–Strebel invariant (Bieri, Neumann, and Strebel, 1987) and its higher degree relatives introduced by Renz (Bieri and Renz, 1988; Renz, 1988, 1989) (the BNSR invariants). Llosa Isenrich and Py (2023) give other constructions of subgroups of Kähler groups with exotic finiteness properties. Certain constructions use morphisms to higher-rank Abelian groups and are not amenable to the strategy we describe below, but rely on the BNSR invariants. Theorem 1.4 in the previous reference constructs subgroups of (not word hyperbolic) Kähler groups with intermediate finiteness properties that are not normal and are themselves Kähler (the construction there involves fiber products rather than morphisms). Dimca, Papadima, and Suciu (2009) constructed the first examples of Kähler groups with intermediate finiteness properties and their techniques (maps to elliptic curves) were pushed further by others; we refer to Llosa Isenrich and Py (2023, section 3.1) for a discussion as well as other references.

The ℓ^2-homology also gives control on the BNSR invariants and on finiteness properties of kernels. A consequence of a theorem of Lück (1998) implies that the kernel of a surjective morphism $G \to \mathbf{Z}$ has not type $\mathrm{FP}_n(\mathbf{Q})$ as soon as the n-th ℓ^2-Betti number of G is nonzero. For the class of residually finite rationally solvable groups (cf. Agol, 2008, for a definition), Kielak (2020, for the case $n = 1$) and Fisher (2022, for the general case) proved that the ℓ^2-Betti numbers of G vanish up to degree n if and only if there is a surjective morphism $G_1 \to \mathbf{Z}$ with kernel of type $\mathrm{FP}_n(\mathbf{Q})$ where G_1 is a finite index subgroup of G. This was involved in the result of Llosa Isenrich, Martelli, and Py (2021) mentioned above.

3. A construction from complex geometry

Hereafter the article Llosa Isenrich and Py (2024) will be mentioned as LlP1 and the article Llosa Isenrich and Py (2023) will be mentioned as LlP2.

In this section we address theorem 1.1. The construction here has three steps. First the kernels of rational cohomology classes of degree 1 coming from complex geometry (precisely admitting a Morse representative that is the real part of a complex differential form with isolated zeros on a Kähler manifold) are shown to produce the wanted example. Second finite-to-one maps to complex tori provide such cohomology classes. Finally some arithmetic quotients of the unit ball in $\mathbf{C}^n$ immerse into their Albanese varieties and thus admit finite-to-one maps to a complex torus. This is the strategy developed in LlP1 with a simplification suggested in LlP2 (section 8) avoiding the use of the BNSR invariants.

3.1. Forms with isolated zeroes

Let X be a compact connected complex manifold. A closed holomorphic 1-form α on X leads to a real differential form $a = \Re\alpha$ that represents an element in the first cohomology group $H^1(X;\mathbf{R})$. When this form is rational, i.e. when the class of a belongs to $H^1(X;\mathbf{Q}) = \mathrm{Hom}(\pi_1(X), \mathbf{Q})$, it gives rise to a homomorphism from $\pi_1(X)$ onto a finitely generated subgroup of $\mathbf{Q}$; hence, up to scaling, it is a surjective homomorphism from $\pi_1(X)$ onto $\mathbf{Z}$. When X is aspherical and α has finitely many zeroes, the kernel of this homomorphism has the desired exotic finiteness properties.

Proposition 3.1 (LlP1, theorem 6.(1)). *Let X be a closed aspherical Kähler manifold of complex dimension $n \geqslant 2$. Let α be a holomorphic 1-form on X with isolated zeroes and let $a = \Re\alpha$. Then there is a neighborhood U of the class of a in $H^1(X;\mathbf{R})$ such that for every b in $U \cap H^1(X;\mathbf{Q})$, the kernel of b is of type $\mathscr{F}_{n-1}$. If furthermore X has nonzero Euler characteristic, then the kernel of b is not of type $\mathrm{FP}_n(\mathbf{Q})$.*

Remark 3.2. Since X is Kähler and closed, holomorphic 1-forms are automatically harmonic and consequently closed. Furthermore, from the Hodge decomposition, the dimension of the space of holomorphic 1-forms is half the first Betti number. Hence the assumption on X is of topological flavor.

A deformation argument (LlP2, section 6.2) shows that the class of a can be represented by a Morse 1-form (i.e. locally the differential of a Morse function) all of whose critical points have index equal to n. This property will hold in a neighborhood U of the class of a in $H^1(X;\mathbf{R})$ (LlP2, proposition 8.1). Let b be a rational form in the open set U and choose β a differential form representing b.

The universal cover $\widetilde{X}$ of X is a contractible manifold and the lift of β is the differential of a function $\widetilde{X} \to \mathbf{R}$. This function descends to a function $f\colon X_0 \to \mathbf{R}$, where $X_0 = \widetilde{X}/\ker b$ is the cover associated with b. The space X_0 is aspherical with fundamental group equal to $\ker b$, thus the finiteness properties of $\ker b$ can be determined from X_0 or from spaces homotopically equivalent to X_0. The function f is proper and has isolated singularities all of index n. Therefore Morse–Lefschetz theory implies that X_0 has the homotopy type of a compact manifold (a regular fiber of f) with infinitely many n-cells attached (as soon as the form α has at least one zero, which is ensured by the assumption on the Euler characteristic). This model for the classifying space of the group $\ker b$ implies that $\ker b$ is indeed of type $\mathscr{F}_{n-1}$. Using a long exact sequence due to Milnor (1968) associated with the cyclic covering $X_0 \to X$, Llosa Isenrich, Martelli, and Py (2021, section 3.2) show that $\ker b$ is not of type $\mathrm{FP}_n(\mathbf{Q})$.