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CONSTRUCTIONS OF NEW EULER SYSTEMS

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1. Introduction

Euler systems are systems of cohomology classes of geometric origin which control the Galois cohomology of Galois representations of automorphic representations. This report recalls briefly the history of the subject (section 2) before giving an overview of three important recent contributions to the construction of new Euler systems: section 3 presents Euler systems for Rankin–Selberg products of modular forms (Bertolini, Darmon, and Rotger, 2015a,b; Darmon and Rotger, 2014, 2017; Kings, Loeffler, and Zerbes, 2020; Lei, Loeffler, and Zerbes, 2014), section 4 the interpolation of zeta elements over universal deformation rings (Nakamura, 2023) and section 5 the use of bipartite Euler systems to prove important cases of the Beilinson–Bloch–Kato conjecture for Rankin–Selberg motives (Liu, Tian, et al., 2022).

1.1. Axiomatic definitions

The first works on Euler systems of V. Kolyvagin and K. Rubin proposed the following tentative axiomatic definitions for the known cases.

Let p be a prime number. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathscr{O}_E$. Let $\Sigma \supset \{p\}$ be a finite set of rational primes. For $K/\mathbb{Q}$ a sub-extension of $\bar{\mathbb{Q}}$, we write $G_{K,\Sigma}$ be the Galois group of the maximal extension of K unramified outside places of K above $\Sigma \cup \{\infty\}$. Now suppose $K/\mathbb{Q}$ is finite. Let (M, ρ) be an E-vector space with a continuous $G_{K,\Sigma}$-action. Because the $G_{K,\Sigma}$-action is continuous and $\mathscr{O}$, we may choose and let (T, ρ) an $\mathscr{O}_E$-lattice inside M which is $G_{K,\Sigma}$-stable. We denote by $M^*(1)$ the dual $G_{K,\Sigma}$-representation $\mathrm{Hom}_E(M, E)(1)$. Let $H^1_f(G_{K,\Sigma}, T)$ be the first Bloch–Kato cohomology group of T (Bloch and Kato, 1990, section 3).

Let Ξ be the set

$$\Xi \overset{\text{def}}{=} \{m \geqslant 1 \mid \{\ell | m\} \cap \Sigma = \{p\}\}. \tag{1}$$

For $m \in \Xi$, we write S_m for the set of primes not in Σ and which divide m (the set Ξ should be understood as indexing partial Euler products; more mundanely its function is to avoid degenerate base cases).

Definition 1.1. Here $K = \mathbb{Q}$. A *cyclotomic Euler system* for T is a system[(1)] of classes

$$\left\{\mathbf{z}(m) \in H^1_f\left(G_{\mathbb{Q}(\zeta_m),\Sigma}, T\right)\right\}_{m\in\Xi} \tag{2}$$

satisfying the following compatibility relation for corestriction

$$\mathrm{Cor}_{\mathbb{Q}(\zeta_{m'})/\mathbb{Q}(\zeta_m)}\,\mathbf{z}(m') = \left(\prod_{\ell\in S_{m'}\setminus S_m} \det\left(1-\mathrm{Fr}(\ell)t|M^*(1)\right)_{t=\sigma(\ell)}\right)\cdot \mathbf{z}(m) \tag{3}$$

for all $m|m' \in \Xi$. Here, ζ_m is a primitive root of unity of order m, $\mathrm{Fr}(\ell)$ is the geometric Frobenius morphism in $G_{\mathbb{Q},\Sigma}$, $\sigma(\ell) \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ is the geometric Frobenius morphism at ℓ and the action on $\mathbf{z}(m)$ in the righthand term of (3) is the natural action of $\mathscr{O}_E[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]$ on $H^1(G_{\mathbb{Q}(\zeta_m),\Sigma}, T)$.

Notice in particular that the definition implies that

$$(\mathbf{z}(mp^n))_{n\geqslant 0} \in \varprojlim_n H^1\left(G_{\mathbb{Q}(\zeta_{mp^n}),\Sigma}, T\right) \simeq H^1\left(G_{\mathbb{Q},\Sigma}, T\otimes_{\mathscr{O}_E}\mathbf{\Lambda}\right) \tag{4}$$

where $\mathbf{\Lambda}$ is the Iwasawa algebra attached to the extension of $\mathbb{Q}(\zeta_{mp^\infty})$.

Fundamental example Let M be the Tate motive $\mathbb{Q}(1)$. Fix a coherent system $\{\zeta_n \in \bar{\mathbb{Q}}|\forall(n,m)\in\mathbb{N}^2, \zeta_{nm}^m = \zeta_n\}_{n\geqslant 1}$ of primitive roots of unity. Put $\Xi = \{p\}$ (this choice is here to ensure that we are avoiding the degenerate case $1-\zeta_n$ for $n=1$ in the following). For all $n \in \Xi$, define the cyclotomic unit

$$\mathbf{z}(n) \overset{\text{def}}{=} (1-\zeta_n)\left(1-\zeta_n^{-1}\right) \in \mathbb{Z}[\zeta_n, 1/\Sigma]^\times \hookrightarrow H^1_f\left(G_{\mathbb{Q}(\zeta_n),\Sigma}, \mathbb{Z}_p(1)\right).$$

Cyclotomic units satisfy the norm relation

$$N_{\mathbb{Q}(\zeta_{m'})/\mathbb{Q}(\zeta_m)}\left((1-\zeta_{m'})\left(1-\zeta_{m'}^{-1}\right)\right) = \left(\prod_{\ell\in S_{m'}\setminus S_m}(1-\sigma(\ell))\right)\left((1-\zeta_n)\left(1-\zeta_m^{-1}\right)\right)$$

where as above $\sigma(\ell)$ is the geometric Frobenius morphism at ℓ in $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. On the other hand, $M^*(1)$ is the trivial motive $\mathbb{Q}$ so $\det\left(1-\mathrm{Fr}(\ell)t|M^*(1)\right) = 1-t$. The classes $\mathbf{z}(m)$ thus satisfy the relation

$$\mathrm{Cor}_{\mathbb{Q}(\zeta_{m'})/\mathbb{Q}(\zeta_m)}\,\mathbf{z}(m') = \left(\prod_{\ell\in S_{m'}\setminus S_m} \det\left(1-\mathrm{Fr}(\ell)t|M^*(1)\right)_{t=\sigma(\ell)}\right)\cdot \mathbf{z}(m).$$

[(1)] Here and in the following, we call the underlying set of classes of an Euler system a *system* in order to emphasize the fact that they are close to being an inverse system for corestriction.

By definition, the system of classes

$$\left\{\mathbf{z}(m) \in H^1_f\left(G_{\mathbb{Q}(\zeta_m),\Sigma}, \mathbb{Z}_p(1)\right)\right\}_{m\in\Xi} \tag{5}$$

is thus a cyclotomic Euler system.

For early accounts of this paradigmatic example of Euler systems and how it can be used to prove bounds on class groups in finite sub-extensions of $\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}$ and Iwasawa Main Conjectures, see (Perrin-Riou, 1990, 1998; Rubin, 2000) among many references.

Definition 1.2. Here $K/\mathbb{Q}$ is a quadratic imaginary extension and T is a self-dual Galois representation. Denote by $K(m)$ the ring-class field of conductor m (an abelian extension of K which is dihedral over $\mathbb{Q}$). An *anticyclotomic Euler system* for T is a system of classes

$$\left\{\mathbf{z}(m) \in H^1_f\left(G_{K(m),\Sigma}, T\right)\right\}_{m\in\Xi} \tag{6}$$

satisfying the following compatibility relation for corestriction

$$\mathrm{Cor}_{K(m\ell)/K(m)}\,\mathbf{z}(m\ell) = \Phi(\ell)\mathbf{z}(m) \tag{7}$$

whenever $\ell \nmid m$ is inert in K (note that this implies that $\ell \notin \Sigma$ unless $\ell = p$), where Φ is an expression linked with the Euler factor of T at ℓ (in important families of early examples, $\Phi(\ell)$ is the trace of $\mathrm{Fr}(\ell)$ the geometric Frobenius morphism at ℓ acting on T).

For early accounts of the typical example of anticyclotomic Euler systems - the Euler system of Heegner points - and how it can be used to prove bounds on Tate–Shafarevich groups, see (Gross, 1991; Perrin-Riou, 1990) among many references.

Because the system of zero classes obviously satisfies the requirements of definitions 1.1 and 1.2, an Euler system is an interesting object only if it contains a non-zero cohomology class, though we resist putting that requirement explicitly in the definitions because proving that an Euler system is non-zero has proven to be one of the hardest problems of the subject.

1.2. Euler systems in arithmetic geometry

The axiomatic definitions above makes no reference to arithmetic or geometry. Indeed, they raise two natural questions.

1. Why should anyone care about Euler systems?
2. Why should non-trivial Euler systems exist?

1.2.1. *What are Euler systems good for?* — The answer to the first question was provided by two groundbreaking insights of KOLYVAGIN (1990). His first contribution was to show that Euler systems gave rise to what has been called Kolyvagin systems after MAZUR and RUBIN (2004), that is to say new systems of cohomology classes whose local behavior forms a cascade of relations: typically, if $(\kappa(m\ell), \kappa(m))$ is a pair of classes in a Kolyvagin system, then the class $\kappa(m)$ is unramified at ℓ, the class $\kappa(m\ell)$ is ramified at ℓ and the localization of $\kappa(m\ell)$ at ℓ may be elucidated in terms of the localization of $\kappa(m)$. More precisely, we have the following definition (see MAZUR and RUBIN (2004, Definition 3.1.3) and HOWARD (2004b, Definition 1.2.3) for details).

Definition 1.3 (HOWARD (2004b), KOLYVAGIN (1990), and MAZUR and RUBIN (2004)). A *cyclotomic Kolyvagin system* for T is a system of classes

$$\left\{\kappa(m) \in H^1_{\mathscr{F}(m)}(G_{\mathbb{Q},\Sigma}, T/I_nT) \otimes G_m\right\}_{m\in\Xi,\text{square-free}} \tag{8}$$

such that

1. $\mathrm{loc}_\ell\, \kappa(m)$ is unramified if $\ell \nmid m$.
2. $\mathrm{loc}_\ell\, \kappa(m\ell)$ belongs to $H^1_s\left(G_{\mathbb{Q}_\ell}, T/I_{m\ell}T\right) \otimes G_{m\ell}$ and

$$\phi^{fs}_\ell(\mathrm{loc}_\ell\, \kappa(m)) = \mathrm{loc}_\ell\, \kappa(m\ell) \in H^1_s\left(G_{\mathbb{Q}_\ell}, T/I_{m\ell}T\right) \otimes G_{m\ell}. \tag{9}$$

   Note that because $m\ell$ is square-free, here also $\ell \nmid m$.

Here I_m is a suitably defined ideal, G_m is the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, the subscripts indicate suitable Galois cohomology conditions and ϕ^{fs}_ℓ is a local comparison map between singular and unramified cohomology.

An *anticyclotomic Kolyvagin system* is defined in the same way for T self-dual, with $\mathbb{Q}$ replaced with a quadratic imaginary extension K, with m a square-free product of inert primes and with G_ℓ replaced with $\mathbb{F}^\times_\lambda/\mathbb{F}^\times_\ell$ for $\lambda \subset \mathscr{O}_K$ the only prime ideal over a rational prime ℓ inert in $\mathscr{O}_K$.

Kolyvagin's second contribution was to show that such a cascade of local relations bounded and often determined fully the Bloch–Kato cohomology groups attached to T. Applying his method to the anticyclotomic system of Heegner points, he obtained the finiteness of the Tate–Shafarevich group of many elliptic curves (KOLYVAGIN, 1990), see section 2 for further details. Simultaneously, Rubin observed that the bounds obtained from Kolyvagin systems were sufficiently uniform in a cyclotomic deformation and would therefore go a long way towards proving Iwasawa Main Conjectures. In this way, he dervied a new proof of the classical Iwasawa Main Conjecture using the cyclotomic Euler system of cyclotomic units and of the Iwasawa Main Conjecture in the $\mathbb{Z}^2_p$-extension of a quadratic imaginary field using the Euler system of of elliptic units (RUBIN, 1991), see section 2 again for further details.

Since then, Euler systems have been used to achieve a variety of remarkable results: proving properties of Mazur–Tate–Teitelbaum $\mathscr{L}$-invariants (Kato, Kurihara, and Tsuji, 1997), showing the finiteness of $\text{Ш}(E/\mathbb{Q})[p^\infty]$ even when E has large algebraic rank (Perrin-Riou, 2003), obtaining parity resuls on the order of vanishing of L-functions (Nekovář, 2013), constructing new p-adic L-functions (Bertolini, Darmon, and Prasanna, 2013), constructing rational points on CM elliptic curves by p-adic methods (Burungale, Kobayashi, and Ota, 2024), etc... They have been an essential component of most of the proofs of Iwasawa Main Conjectures and Equivariant Tamagawa Number Conjectures known to date. This amply justifies the quest for new Euler systems.

In this survey, we henceforth concentrate on how to construct them.

1.2.2. *Why do Euler systems exist?* — The second question has been the object of considerable interest ever since the first seminal works on Euler systems by Flach (1992), Kolyvagin (1990, 1991b), Kolyvagin and Logachëv (1991), Nekovář (1992), Rubin (1991), and Thaine (1988). For instance, Kato (1993b) made the following remark in his inimitable style.

> It seems to me that only known general method to discover such important elements is to open our mouths and wait for such elements to drop from the sky. However l do not know why these people[(2)] with small mouths can catch such elements so often.

An ambition of this report is to explain that multiple answers have emerged after developments of the last two decades, beyond waiting for such elements to drop from the sky.

As recalled in the brief historical section below, the chronologically first answer is due to Kato himself, who pointed out in Kato (1993a) that if the conjectures of Bloch–Kato on special values of L-functions of motives (Bloch and Kato, 1990) were true for motives with coefficients in $\mathbb{Q}[\mathrm{Gal}(K/\mathbb{Q})]$ and compatible with changes of extension (what has been called the equivariant refinement), then systems of classes satisfying the norm relations of definition 1.1 should exist in a very broad sense. Interestingly, these class do not live in $H^1_f(G_{\mathbb{Q}(\zeta_m),\Sigma}, T)$ in general but in the so-called p-adic fundamental line of the Galois representation M, that is to say the E-vector space of dimension 1

$$\mathrm{Det}_E^{-1}\,\mathrm{R}\,\Gamma_{\mathrm{et}}\,(\mathbb{Z}[\zeta_m, 1/p], M) \otimes_E \mathrm{Det}_E^{-1}\, M(-1)^+. \tag{10}$$

Here $(-)^+$ means the subspace invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, Det_A denotes the determinant functor from the category of perfect complexes of projective A-modules to the category of graded invertible A-modules (Knudsen and Mumford, 1976) and $\mathrm{R}\,\Gamma_{\mathrm{et}}\,(\mathbb{Z}[\zeta_m, 1/p], -)$ is the étale cohomology complex functor $\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{Q}(\zeta_m), j_*-)$ j for the morphism $j\colon\ \mathrm{Spec}\,\mathbb{Q}(\zeta_m) \longrightarrow \mathrm{Spec}\,\mathbb{Z}[\zeta_m]$. In important early examples

(2) That is to say S. Bloch and A. Beilinson.