Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 152
Fascicule 4

2024

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 785-855

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 4, tome 152, décembre 2024

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 160 € ($ 240),
avec supplément papier : Europe 244 €, hors Europe 330 € ($ 421)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
152 (4), 2024, p. 785-855

COMPARING BUSHNELL–KUTZKO AND SÉCHERRE'S CONSTRUCTIONS OF TYPES FOR GL_N AND ITS INNER FORMS WITH YU'S CONSTRUCTION

by Arnaud Mayeux & Yuki Yamamoto

Dedicated to Colin J. Bushnell

Abstract. — Let F be a non-Archimedean local field with odd residual characteristic, A be a central simple F-algebra, and G be the multiplicative group of A. To construct types for complex supercuspidal representations of G, simple types by Sécherre and Yu's construction are already known. In this paper, we compare these constructions. In particular, we show essentially tame supercuspidal representations of G defined by Bushnell–Henniart are nothing but tame supercuspidal representations defined by Yu.

Texte reçu le 24 janvier 2022, modifié le 15 mai 2024, accepté le 18 juin 2024.

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Mathematical subject classification (2010). — 11F70, 11S37, 20G05, 20G25, 22E50.

Key words and phrases. — Local Langlands correspondence, representation theory of p-adic groups, explicit constructions of supercuspidal representations, Bushnell–Kutzko type theory, Yu's construction, Bushnell–Kutzko's construction, Sécherre's construction, tamely ramified representations, wildly ramified representations, inner forms of GL_N, Howe factorizations, simple characters, Moy–Prasad filtrations, Moy–Prasad isomorphism, Moy–Prasad depth, Bruhat–Tits buildings.

Arnaud Mayeux was supported by ISF grant 1577/23. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (A.M. grant agreement No 101002592 (2021-2023)). The first author was previously funded by a Boya postdoctoral fellowship of BICMR and Peking University (2019–2021) and a doctoral fellowship of Université Paris Cité (2015-2019). The second author is supported by the FMSP program at Graduate School of Mathematical Sciences, the University of Tokyo. He was also supported by JSPS KAKENHI Grant Number JP21J13751.

doi:10.24033/bsmf.1524amyy

Résumé (*Comparer les constructions de Bushnell–Kutzko et Sécherre de types pour GL_N et ses formes intérieurs à la construction de Yu*). — Soit F un corps local non-archimédien de caractéristique résiduelle impaire, A une F-algèbre centrale simple, et G le groupe multiplicatif de A. Afin de construire des types pour les représentations supercuspidales complexes de G, on dispose des constructions de Sécherre et Yu. Dans cet article, nous comparons ces constructions. En particulier, nous montrons que les représentations essentiellement modérées introduites par Bushnell-Henniart sont des représentations modérées de Yu.

1. Introduction

Let F be a non-Archimedean local field such that the residual characteristic p is odd, and let G be the group of F-points of a connected reductive group defined over F. The aim of type theory is to classify, up to some natural equivalence, the smooth irreducible complex representations of G in terms of representations of compact open subgroups. For complex supercuspidal representations of G, some constructions of types are known.

For example, Bushnell–Kutzko [7] constructed types, called simple types, for any irreducible supercuspidal representations when $G = \mathrm{GL}_N(F)$. Sécherre [26], [27], [28], and Sécherre–Stevens [29] extended the construction of simple types to any irreducible supercuspidal representations of any inner form G of $\mathrm{GL}_N(F)$.

For an arbitrary reductive group G, Yu's construction [31] (cf. also [1] for a similar pioneering method) provides some supercuspidal representations. In his paper [31, p580], Yu wrote "*In particular, our method should yield all supercuspidal representations when p is large enough relative to the type of G.*", Yu also wrote "*it is possible that our method yields all supercuspidal representations that deserve to be called tame*". Yu's expectations are now theorems by works of Kim [19] and Fintzen [12]. More precisely, Yu's construction yields all supercuspidal representations if p does not divide the order of the Weyl group of G by [12], a condition that guarantees that all tori of G split over a tamely ramified field extension of F.

It is a natural question whether there exists some relationship between these constructions of types. A natural motivation being to unify or generalize these constructions by taking advantage of each theory. In his paper [31, p581], Yu wrote "*However, the real difficulty in the wild case is that considerably different* (authors note: than his) *constructions should be involved as revealed in the GL_n case by the work of Bushnell, Corwin, and Kutzko.*" The goal of the present article is to carefully compare Bushnell–Kutzko and Sécherre's constructions to Yu's one.

From now on, let A be a finite dimensional central simple F-algebra, and let V be a simple left A-module. Then $\mathrm{End}_A(V)$ is a central division F-algebra.

Let D be the opposite algebra of $\mathrm{End}_A(V)$. Then V is also a right D-module, and we have $A \cong \mathrm{End}_D(V)$. Let G be the multiplicative group of A. Then we have $G \cong \mathrm{GL}_m(D)$, which is the group of F-points of an inner form of GL_N.

We introduce our main theorem. In Yu's construction, from a tuple $\Psi = (x, (G^i)_{i=0}^d, (\mathbf{r}_i)_{i=0}^d, (\mathbf{\Phi}_d)_{i=0}^d, \rho)$, which is called a Yu datum of G, one constructs some open subgroups ${}^\circ K^d(\Psi)$, $K^d(\Psi)$ in G and an irreducible representation $\rho_d(\Psi)$ of $K^d(\Psi)$, which are precisely defined in §3.

Theorem 1.1 (Theorem 10.6). — *Let (J, λ) be a simple type for an essentially tame supercuspidal representation π, and let $(\tilde{J}, \Lambda)$ be an extension of (J, λ) such that $\pi \cong \text{c-Ind}_{\tilde{J}}^G \Lambda$. Then there exists a Yu datum $\Psi = (x, (G^i)_{i=0}^d, (\mathbf{r}_i)_{i=0}^d, (\mathbf{\Phi}_d)_{i=0}^d, \rho)$ satisfying the following conditions:*

1. *$J = {}^\circ K^d(\Psi)$.*
2. *$\tilde{J} \subset K^d(\Psi)$.*
3. *$\rho_d(\Psi) \cong \text{c-Ind}_{\tilde{J}}^{K^d(\Psi)} \Lambda$.*

Theorem 1.2 (Theorem 11.8). — *Conversely, let $\Psi = \big(x, (G^i), (\mathbf{r}_i), (\mathbf{\Phi}_i), \rho\big)$ be a Yu datum of G. Then there exists a tame simple type (J, λ) and a maximal extension $(\tilde{J}, \Lambda)$ of (J, λ) such that we have the following.*

1. *${}^\circ K^d(\Psi) = J$.*
2. *$K^d(\Psi) \supset \tilde{J}$.*
3. *$\rho_d(\Psi) \cong \text{c-Ind}_{\tilde{J}}^{K^d(\Psi)} \Lambda$.*

By these theorems, we obtain the following corollary.

Corollary 1.3 (Corollary 11.9). — *For any inner form G of $\mathrm{GL}_N(F)$, the set of essentially tame supercuspidal representations of G is equal to the set of tame supercuspidal representations of G defined by Yu* [31].

In particular, for $G = \mathrm{GL}_N(F)$, the statements of the above theorems are as follows:.

Theorem 1.4. — *Let $G = \mathrm{GL}_N(F)$. Then $\tilde{J} = K^d(\Psi)$ in the statement of (2) in Theorem 1.1 and Theorem 1.2, and $\rho_d(\Psi) \cong \Lambda$ in the statement of (3) in these theorems.*

We sketch the outline of this paper and the main steps to prove Theorems 1.1 and 1.2. First, in §2 and 3, we recall constructions of types. We explain simple types of G by Sécherre in §2 and Yu's construction of tame supercuspidal representations in §3. Next, in §4-9, we prepare ingredients to compare these two constructions. A class of simple types corresponding to Yu's type is defined in §4. In §5, we determine tame twisted Levi subgroups in G. For some tame twisted Levi subgroup G' in G and some "nice" x in the enlarged Bruhat–Tits building $\mathscr{B}^E(G', F)$, we obtain another description of Moy–Prasad filtration on $G'(F)$ attached to x, using hereditary orders, in §6. Then we can compare the groups that the types are defined as representations of. In §7, we discuss generic elements and generic characters. We relate them to some defining sequence of some simple stratum. In §8, we show some lemmas on simple types of depth zero. These lemmas are used to take "depth-zero" parts of types. In §9, we represent a simple character with a tame simple stratum as a product of characters. This factorization is needed to construct generic characters. Finally, in §10 and 11, we prove the main theorem. In §9, from tame Sécherre data, which are used to construct tame simple types, we construct Yu data. By comparing these types constructed by these data and confirming some kind of match between the two, we show that tame simple types can be constructed from Yu's types. Conversely, we also show that Yu's types are constructed from tame simple types in §11. In §12, we briefly discuss the wild case.

Remark 1.5. — The first version of this work, containing all main ideas and arguments, appeared publicly in June 2017. This brings together and extends previous works of the authors that are not intended to be published in journals. These works are precisely:

- A. Mayeux: *Représentations supercuspidales: comparaison des construction de Bushnell–Kutzko et Yu*, arXiv:1706.05920, 2017, unpublished.
- A. Mayeux: *Comparison of Bushnell–Kutzko and Yu's constructions of supercuspidal representations*, arXiv:2001.06259, 2020, unpublished.
- Y. Yamamoto: *Comparison of types for inner forms of GL_N*, arXiv: 2005.02622, 2020, unpublished.
- The parts about the comparison of our PhD theses defended at Paris in 2019 and Tokyo in 2022.

The present paper covers all the mathematical content of all unpublished works listed above. Our present paper is mathematically self-contained in the sense that it does not rely on the previously mentioned unpublished works.

Notation. — If X is a scheme over a base scheme S and if $T \to S$ is a morphism of schemes, then X_T denotes $X \times_S T$ and is seen as a scheme over T.

Let $G \to S$ be a group scheme acting on a scheme X/S. The functor of fixed points, by definition, sends a scheme T over S to $\{x \in \mathrm{Hom}_T(T, X_T) | x \text{ is } G_T\text{-equivariant}\}$, where T is endowed with the trivial action of G_T. This S-functor is denoted by X^G. Note that for any scheme T over S, we have $X^G(T) \subset X(T)$. It is known that this functor is representable by a scheme in many cases (cf., e.g., [9, Exp. 12 Prop. 9.2]).

In this paper, we consider smooth representations over $\mathbb{C}$. We fix a non-Archimedean local field F such that the residual characteristic p is odd. For a finite-dimensional central division algebra D over F, let $\mathfrak{o}_D$ be the ring of integers, $\mathfrak{p}_D$ be the maximal ideal of $\mathfrak{o}_D$, and let k_D be the residual field of D. We fix a smooth, additive character $\psi : F \to \mathbb{C}^\times$ of conductor $\mathfrak{p}_F$. For a finite field extension E/F, let v_E be the unique surjective valuation $E \to \mathbb{Z} \cup \{\infty\}$.

Moreover, for any element β in some algebraic extension field of F, we put $\mathrm{ord}(\beta) = e(F[\beta]/F)^{-1} v_{F[\beta]}(\beta)$.

If K is a field and G is a K-group scheme, then $\underline{\mathrm{Lie}}(G)$ denotes the Lie algebra functor, and we put $\mathrm{Lie}(G) = \underline{\mathrm{Lie}}(G)(K)$. If a K-group scheme is denoted by a capital letter G, the Lie algebra functor of G is denoted by the same small Gothic letter $\mathfrak{g}$. We also denote by $\mathrm{Lie}^*(G)$ or $\mathfrak{g}^*(K)$ the dual of $\mathrm{Lie}(G) = \mathfrak{g}(K)$. For connected reductive group G over F, we denote by $\mathscr{B}^E(G,F)$ (resp. $\mathscr{B}^R(G,F)$) the enlarged Bruhat–Tits building (resp. the reduced Bruhat–Tis building) of G over F defined in [4], [5]. If $x \in \mathscr{B}^E(G,F)$, we denote by $[x]$ the image of x via the canonical surjection $\mathscr{B}^E(G,F) \to \mathscr{B}^R(G,F)$. The group $G(F)$ acts on $\mathscr{B}^E(G,F)$ and $\mathscr{B}^R(G,F)$. For $x \in \mathscr{B}^E(G,F)$, let $G(F)_x$ (reps. $G(F)_{[x]}$) denote the stabilizer of $x \in \mathscr{B}^E(G,F)$ (resp. $[x] \in \mathscr{B}^R(G,F)$). We denote by $\tilde{\mathbb{R}}$ the totally ordered commutative monoid $\mathbb{R} \cup \{r+ \mid r \in \mathbb{R}\}$. When G splits over some tamely ramified extension of F, for $x \in \mathscr{B}^E(G,F)$ let $\{G(F)_{x,r}\}_{r \in \tilde{\mathbb{R}}_{\geq 0}}$, $\{\mathfrak{g}(F)_{x,r}\}_{r \in \tilde{\mathbb{R}}}$ and $\{\mathfrak{g}^*(F)_{x,r}\}_{r \in \tilde{\mathbb{R}}}$ be the Moy–Prasad filtration [23], [24] on $G(F)$, $\mathfrak{g}(F)$ and $\mathfrak{g}^*(F)$, respectively. Here, we have $\mathfrak{g}^*(F)_{x,r} = \{X^* \in \mathfrak{g}^*(F) \mid X^*(\mathfrak{g}(F)_{x,(-r)+}) \subset \mathfrak{p}_F\}$ for $r \in \mathbb{R}$. If G is a torus, Moy–Prasad filtrations are independent of x, and then we omit x.

Let G be a group, H be a subgroup in G and λ be a representation of H. Then we put ${}^gH = gHg^{-1}$ for $g \in G$ and we define a ${}^gH$-representation ${}^g\lambda$ as ${}^g\lambda(h) = \lambda(g^{-1}hg)$ for $h \in {}^gH$. Moreover, we also put

$$I_G(\lambda) = \{g \in G \mid \mathrm{Hom}_{H \cap {}^gH}(\lambda, {}^g\lambda) \neq 0\}.$$

2. Simple types by Sécherre

We recall the theory of simple types from [26], [27], [28], [29]. In this section, we can omit the assumption that the residual characteristic of F is odd.

2.1. Lattices, hereditary orders. — Let D be a finite-dimensional central division F-algebra. Let V be a right D-module with $\dim_F V < \infty$. We put $A = \mathrm{End}_D(V)$, and then A is a central simple F-algebra. Moreover, there exists $m \in \mathbb{Z}_{>0}$ such that $A \cong \mathrm{M}_m(D)$. Let G be the multiplicative group of A, and then G is isomorphic to $\mathrm{GL}_m(D)$. We also put $d = (\dim_F D)^{1/2}$ and $N = md$.

An $\mathfrak{o}_D$-submodule $\mathcal{L}$ in V is called a lattice if and only if $\mathcal{L}$ is a compact open submodule.

Definition 2.1 ([26, Définition 1.1]). — For $i \in \mathbb{Z}$, let $\mathcal{L}_i$ be a lattice in V. We say that $\mathcal{L} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$ is an $\mathfrak{o}_D$-sequence if

1. $\mathcal{L}_i \supset \mathcal{L}_j$ for any $i < j$, and
2. there exists $e \in \mathbb{Z}_{>0}$ that $\mathcal{L}_{i+e} = \mathcal{L}_i \mathfrak{p}_D$ for any i.