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DIAGONAL CYCLES AND EULER SYSTEMS I: A p-ADIC GROSS-ZAGIER FORMULA

BY HENRI DARMON AND VICTOR ROTGER

ABSTRACT. – This article is the first in a series devoted to studying *generalised Gross-Kudla-Schoen diagonal cycles* in the product of three Kuga-Sato varieties and the Euler system properties of the associated Selmer classes, with special emphasis on their application to the Birch-Swinnerton-Dyer conjecture and the theory of Stark-Heegner points. The basis for the entire study is a p-adic formula of Gross-Zagier type which relates the images of these diagonal cycles under the p-adic Abel-Jacobi map to special values of certain p-adic L-functions attached to the Garrett-Rankin triple convolution of three Hida families of modular forms. The main goal of this article is to describe and prove this formula.

RÉSUMÉ. – Cet article est le premier d'une série consacrée aux *cycles de Gross-Kudla-Schoen généralisés* appartenant aux groupes de Chow de produits de trois variétés de Kuga-Sato, et aux *systèmes d'Euler* qui leur sont associés. La série au complet repose sur une variante p-adique de la formule de Gross-Zagier qui relie l'image des cycles de Gross-Kudla-Schoen par l'application d'Abel-Jacobi p-adique aux valeurs spéciales de certaines fonctions L p-adiques attachées à la convolution de Garrett-Rankin de trois familles de Hida de formes modulaires cuspidales. L'objectif principal de cet article est de décrire et de démontrer cette variante.

1. Introduction

This article is the first in a series devoted to studying *generalized diagonal cycles* in the product of three Kuga-Sato varieties and the Euler system properties of the associated Selmer classes, with special emphasis on their application to the Birch–Swinnerton-Dyer conjecture and the theory of Stark-Heegner points. The basis for the entire study is a p-adic Gross-Zagier formula relating

- the image under the p-adic Abel-Jacobi map of certain *generalized Gross-Kudla-Schoen cycles* in the product of three Kuga-Sato varieties, to
- the special value of the p-adic L-function of [19] attached to the Garrett-Rankin triple convolution of three Hida families of modular forms, at a point lying outside its region of interpolation.

In order to precisely state the main result, let

$$f = \sum a_n(f)q^n \in S_k(N_f, \chi_f),$$
$$g = \sum a_n(g)q^n \in S_\ell(N_g, \chi_g),$$
$$h = \sum a_n(h)q^n \in S_m(N_h, \chi_h)$$

be three normalized primitive cuspidal eigenforms of weights k, ℓ, $m \geq 2$, levels N_f, N_g, $N_h \geq 1$, and Nebentypus characters χ_f, χ_g, and χ_h, respectively. Let $N := \mathrm{lcm}(N_f, N_g, N_h)$ and assume that

$$\chi_f \cdot \chi_g \cdot \chi_h = 1,$$

so that in particular $k + \ell + m$ is even.

The triple (k, ℓ, m) is said to be *balanced* if the largest weight is strictly smaller than the sum of the other two. A triple of weights which is not balanced will be called *unbalanced*, and the largest weight in an unbalanced triple will be referred to as the *dominant weight*.

Section 4.1 recalls the definition of the Garrett-Rankin L-function $L(f, g, h; s)$ attached to the triple tensor product

$$V_p(f, g, h) := V_p(f) \otimes V_p(g) \otimes V_p(h)$$

of the (compatible systems of) p-adic Galois representations $V_p(f)$, $V_p(g)$ and $V_p(h)$ attached to f, g and h respectively. This L-function satisfies a functional equation relating its values at s and $k + \ell + m - 2 - s$. In particular, the parity of the order of vanishing of $L(f, g, h; s)$ at the central critical point $c := \frac{k+\ell+m-2}{2}$ is controlled by the sign $\varepsilon \in \{\pm 1\}$ in this functional equation, a quantity that can be expressed as a product $\varepsilon = \prod_{v|N\infty} \varepsilon_v$, $\varepsilon_v \in \{\pm 1\}$, of local root numbers indexed by the places dividing $N\infty$. The following hypothesis is assumed throughout:

H: The local root numbers ε_v at all the finite primes $v \mid N$ are equal to $+1$.

This assumption holds in a broad collection of settings of arithmetic interest. For instance, it is satisfied in either of the following two cases:

- $\gcd(N_f, N_g, N_h) = 1$, or,
- $N = N_f = N_g = N_h$ is square-free and $a_v(f)a_v(g)a_v(h) = -1$ for all primes $v \mid N$.

Assumption H implies that $\varepsilon = \varepsilon_\infty$ depends only on the local sign at ∞, which in turn depends only on whether the weights of (f, g, h) are balanced or not:

$$\varepsilon = \varepsilon_\infty = \begin{cases} -1 & \text{if } (k, \ell, m) \text{ is balanced;} \\ \ \ 1 & \text{if } (k, \ell, m) \text{ is unbalanced.} \end{cases}$$

In particular, the L-function $L(f, g, h, s)$ necessarily vanishes (to odd order) at its central point c when (k, ℓ, m) is balanced.

Let $\mathcal{E}$ denote the universal generalized elliptic curve fibered over $X = X_1(N)$. For any $n \geq 0$, let $\mathcal{E}^n$ be the n-th Kuga-Sato variety over $X_1(N)$. It is an $n + 1$-dimensional variety obtained by desingularising the n-fold fiber product of $\mathcal{E}$ over $X_1(N)$. (Cf. [35] for a more detailed account of its construction.) The p-adic Galois representation $V_p(f, g, h)$ occurs in the middle cohomology of the triple product

$$W := \mathcal{E}^{k-2} \times \mathcal{E}^{\ell-2} \times \mathcal{E}^{m-2}. \qquad (1.1)$$

When (k, ℓ, m) is balanced and assumption H is satisfied, the conjectures of Bloch-Kato and Beilinson-Bloch predict (because of the vanishing of $L(f, g, h, c)$) that there should then exist a non-trivial cycle in the Chow group $\mathbb{Q} \otimes \mathrm{CH}^c(W)_0$ of rational equivalence classes of null-homologous cycles of codimension c on the variety W of (1.1). Section 3.1 introduces cycles $\Delta_{f,g,h} \in \mathbb{Q} \otimes \mathrm{CH}^c(W)_0$ which are natural candidates to fulfill these expectations, and whose construction we now briefly summarize.

Set $r = \frac{k+\ell+m-6}{2}$. As explained in §3.1, there exists an essentially unique, natural way of embedding the Kuga-Sato variety $\mathcal{E}^r$ in the variety W. Its image gives rise to an element in the Chow group $\mathrm{CH}^{r+2}(W)$ which, suitably modified, becomes homologically trivial. In this way, we obtain a cycle

$$\Delta_{k,\ell,m} \in \mathrm{CH}^{r+2}(W)_0 := \ker(\mathrm{CH}^{r+2}(W) \xrightarrow{\mathrm{cl}} H^{2r+4}_{\mathrm{dR}}(W/\mathbb{C})).$$

In the special case where $k = \ell = m = 2$, the cycle $\Delta_{2,2,2}$ is just the modified diagonal considered by Gross–Kudla [15] and Gross–Schoen [17].

The cycles $\Delta_{f,g,h}$ alluded to above are defined as the (f, g, h)-isotypical component of the null-homologous cycle $\Delta_{k,\ell,m}$ with respect to the action of the Hecke operators.

It is natural to conjecture that the heights of these cycles in the sense of Beilinson and Bloch are well-defined (cf. [15] and [17] for more details on the necessary definitions), and can be directly related to the first derivative of the triple product L-function $L(f, g, h, s)$ at the central point:

$$h(\Delta_{f,g,h}) \stackrel{?}{=} (\text{Explicit non-zero factor}) \times L'(f, g, h, r + 2). \tag{1.2}$$

When $(k, \ell, m) = (2, 2, 2)$, this was predicted in [15] and has recently been proved by X. Yuan, S. Zhang and W. Zhang in [40].

REMARK 1.1. – It would be natural to relax assumption H to the weaker condition

(1.3) H_{even}: The set of primes $v \mid N$ for which $\varepsilon_v = -1$ is of *even cardinality*.

This is sufficient to guarantee that $\varepsilon = \varepsilon_\infty$, and can be dealt with at the cost of replacing Kuga-Sato varieties with more general objects arising from the self-fold products of certain families of abelian surfaces (or genus two curves) fibered over Shimura curves rather than classical modular curves. Hypothesis H may thus be regarded as analogous to the classical Heegner or Gross-Zagier hypothesis imposed in the study of the Rankin-Selberg L-function $L(f \otimes \theta_K, s)$ attached to a single eigenform f and the weight one theta series of an imaginary quadratic field K. Both are meant to avoid having to deal with Shimura curves associated with a quaternion division algebra, and make it possible to confine one's attention to classical modular curves. Much of our study extends to the setting of H_{even} by appealing to the work of P. Kassaei [25] and R. Brasca [6]; in our exposition we have tried to present our results in a way that suggests the modifications necessary to deal with arbitrary Shimura curves.

In this work we do not focus on (1.2), but rather on a p-adic analogue. Our main result relates the image of $\Delta_{f,g,h}$ under the p-adic Abel-Jacobi map

$$\mathrm{AJ}_p : \mathrm{CH}^{r+2}(W)_0(\mathbb{Q}_p) \longrightarrow \mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(W/\mathbb{Q}_p)^\vee \tag{1.4}$$

to the special value of a triple product p-adic L-function attached to three Hida families of modular forms, which we now describe in more detail.

Fix an odd prime number $p \nmid N$ at which f, g and h are ordinary. Let

$$\mathbf{f} : \Omega_f \longrightarrow \mathbb{C}_p[[q]], \qquad \mathbf{g} : \Omega_g \longrightarrow \mathbb{C}_p[[q]], \quad \mathbf{h} : \Omega_h \longrightarrow \mathbb{C}_p[[q]]$$

denote the Hida families of overconvergent p-adic modular forms passing though f, g and h, respectively, as constructed in [21] and [20], and briefly reviewed in §2.6 below. The spaces Ω_f, Ω_g and Ω_h are finite rigid analytic coverings of suitable subsets of the *weight space*

$$\Omega := \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times),$$

which contains the integers $\mathbb{Z}$ as a dense subset via the natural inclusion $k \mapsto (x \mapsto x^k)$. A point $x \in \Omega_f$ is said to be *classical* if its image in Ω, denoted $\kappa(x)$, belongs to $\mathbb{Z}^{\geq 2}$, and the set of classical points in Ω_f is denoted by $\Omega_{f,\mathrm{cl}}$. Part of the requirement that $\mathbf{f}$ be a Hida family is that the formal q-series $f_x^{(p)} := \mathbf{f}(x)$ should correspond to a normalized eigenform of weight $\kappa(x)$ on $\Gamma_1(N) \cap \Gamma_0(p)$, for almost all $x \in \Omega_{f,\mathrm{cl}}$. For all but finitely many such x, the form $f_x^{(p)}$ is the ordinary p-stabilization of a normalized eigenform on $\Gamma_1(N)$, denoted f_x.

The natural domain of definition of the triple product p-adic L-functions is the p-adic analytic space

$$\Sigma := \Omega_f \times \Omega_g \times \Omega_h.$$

Let $\Sigma_{\mathrm{cl}} := \Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}} \subset \Sigma$ denote its subset of "classical points". This set is naturally partitioned into four disjoint subsets:

$$\begin{aligned}
\Sigma_f &= \{(x,y,z) \in \Sigma_{\mathrm{cl}} \text{ such that } \kappa(x) \geq \kappa(y) + \kappa(z)\};\\
\Sigma_g &= \{(x,y,z) \in \Sigma_{\mathrm{cl}} \text{ such that } \kappa(y) \geq \kappa(x) + \kappa(z)\};\\
\Sigma_h &= \{(x,y,z) \in \Sigma_{\mathrm{cl}} \text{ such that } \kappa(z) \geq \kappa(x) + \kappa(y)\};\\
\Sigma_{\mathrm{bal}} &= \{(x,y,z) \in \Sigma_{\mathrm{cl}} \text{ such that } (\kappa(x), \kappa(y), \kappa(z)) \text{ is balanced}\}.
\end{aligned}$$

Section 4 exploits the strategy pioneered by Hida [22] and subsequently extended by Harris and Tilouine [19] to construct *three a priori distinct* p-adic L-functions of three variables, denoted

$$\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h}), \quad \mathscr{L}_p{}^g(\mathbf{f},\mathbf{g},\mathbf{h}), \quad \mathscr{L}_p{}^h(\mathbf{f},\mathbf{g},\mathbf{h}) : \Sigma \longrightarrow \mathbb{C}_p,$$

which interpolate the *square-roots* of the *central critical values* of the classical L-function $L(f_x, g_y, h_z, s)$, as (x,y,z) ranges over Σ_f, Σ_g, and Σ_h respectively. The precise interpolation property defining the three p-adic L-functions is spelled out in Theorem 4.7 of Section 4.2.

Given $(x,y,z) \in \Sigma_{\mathrm{bal}}$, the Heegner assumption H can be used to show that the classical L-function $L(f_x, g_y, h_z, s)$ vanishes at its central point for reasons of sign. The *central critical derivative* $L'(f_x, g_y, h_z, \frac{\kappa(x)+\kappa(y)+\kappa(z)-2}{2})$ is then a natural object of arithmetic interest. In the p-adic realm, the three distinct p-adic avatars of the classical L-function, namely, $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})$, $\mathscr{L}_p{}^g(\mathbf{f},\mathbf{g},\mathbf{h})$, and $\mathscr{L}_p{}^h(\mathbf{f},\mathbf{g},\mathbf{h})$, need not vanish at the balanced point (x,y,z), since this point lies outside the region of classical interpolation. The corresponding p-adic special values can be viewed as different p-adic avatars of the complex leading term, and one might expect them to encode similar information related to the motive of $V_{f_x} \otimes V_{g_y} \otimes V_{h_z}$.