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THE CLASSIFICATION OF POLYNOMIAL BASINS OF INFINITY

BY LAURA DEMARCO AND KEVIN PILGRIM

ABSTRACT. – We consider the problem of classifying the dynamics of complex polynomials $f : \mathbb{C} \to \mathbb{C}$ restricted to the basins of infinity $X(f)$. We synthesize existing combinatorial tools—tableaux, trees, and laminations—into a new invariant of basin dynamics we call the pictograph. For polynomials with all critical points escaping to infinity, we obtain a complete description of the set of topological conjugacy classes with given pictograph. For arbitrary polynomials, we compute the total number of topological conjugacy classes of basins $(f, X(f))$ with a given pictograph. We also define abstract pictographs and prove that every abstract pictograph is realized by a polynomial. Extra details are given in degree 3, and we provide examples that show the pictograph is a finer invariant than both the tableau of [5] and the tree of [10].

RÉSUMÉ. – Nous étudions la question de la classification de la dynamique des polynômes complexes $f : \mathbb{C} \to \mathbb{C}$ restreints à leur bassin de l'infini. Nous faisons la synthèse d'outils de combinatoire — tableaux, arbres, laminations — en un nouvel invariant du bassin dynamique que nous appelons *pictogramme*. Pour les polynômes dont tous les points critiques s'échappent vers l'infini, nous obtenons une description complète de l'ensemble des classes de conjugaison topologiques ayant un pictogramme donné. Plus généralement, pour tout polynôme, nous calculons le nombre de classes de conjugaison topologiques du bassin $(f, X(f))$ à pictogramme donné. Nous définissons les pictogrammes de façon abstraite et prouvons que chacun d'eux est réalisable par un polynôme. Nous donnons plus de détails en degré 3 et donnons des exemples montrant que le pictogramme est un invariant plus fin que les tableaux de [5] et que les arbres de [10].

1. Introduction

This article continues a study of the moduli space of complex polynomials $f : \mathbb{C} \to \mathbb{C}$, in each degree $d \geq 2$, in terms of the dynamics of polynomials on their basins of infinity [4, 5, 10, 8, 7]. Our main goal is to classify the topological conjugacy classes of a polynomial f restricted to its basin

$$X(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}.$$

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The basin $X(f)$ is an open, connected subset of $\mathbb{C}$. In degree $d = 2$, there are only two topological conjugacy classes of basins $(f, X(f))$, distinguished by the Julia set being connected or disconnected; see, for example, [19, Theorem 10.1]. In every degree $d > 2$, there are infinitely many topological conjugacy classes of basins, even among the structurally stable polynomials in the shift locus. The main objective of this article is the development of combinatorial methods that allow us to distinguish and enumerate these conjugacy classes in all degrees.

By definition, a polynomial f of degree d is in the shift locus if all of its $d-1$ critical points are in $X(f)$. In this case, the basin $X(f)$ is a rigid Riemann surface, admitting up to Möbius transformations a unique embedding into the Riemann sphere ([18, §2.8], [1, §IV.4]). In this case, the restriction $f : X(f) \to X(f)$ uniquely determines the conformal conjugacy class of $f : \mathbb{C} \to \mathbb{C}$. Thus, our results on basin dynamical systems $(f, X(f))$—given as Theorems 1.1, 1.2, 1.3, and 1.4 below—also provide a combinatorial classification of topological conjugacy classes of polynomials in the shift locus.

In the theory of dynamical systems, the study of a system like $(f, X(f))$ is somewhat nonstandard. On the one hand, since all points tend to ∞ under iteration, the system is transient. On the other hand, the structure of $(f, X(f))$, with an induced dynamical system on its Cantor set of ends as a topological space, carries enough information to recover the full entropy of the polynomial $(f, \mathbb{C})$; see [10, Theorem 1.1]. Our methods and perspective are inspired by the two foundational articles of Branner and Hubbard on polynomial dynamics which lay the groundwork and treat the case of cubic polynomials in detail [4, 5].

1.1. The pictograph, informally described

We begin with a rough description of the pictograph $\mathscr{D}(f)$ associated to a polynomial basin dynamics $(f, X(f))$. A formal presentation is given in §2.2 (for cubic polynomials) and Section 10 (in arbitrary degree).

Global setup. – The basin dynamics $f : X(f) \to X(f)$ fits naturally into a sequence of dynamical systems related by semiconjugacies. These are organized in the diagram below, and are explained in the following paragraphs.

$$\mathcal{X}(f) \circlearrowleft \mathscr{I} \xrightarrow{g_f} X(f) \circlearrowleft f \xrightarrow{\pi_f} T(f) \circlearrowleft F \xrightarrow{h_f} (0,\infty) \circlearrowleft \cdot d, \qquad G_f = h_f \circ \pi_f : X(f) \to (0,\infty). \tag{1.1}$$

The map G_f is the harmonic *Green's function*; its values we call *heights* or sometimes *escape rates*. The grand orbits (under multiplication by d) of heights of critical points are called *nongeneric heights*. We endow $(0, \infty)$ with a simplicial structure in which the nongeneric heights are vertices. The map π_f collapses connected components of level sets of G_f to points; its image is the *DeMarco-McMullen tree* $T(f)$, and f induces a self-map $F : T(f) \to T(f)$. By construction, the factor $h_f : T(f) \to (0, \infty)$ is simplicial.

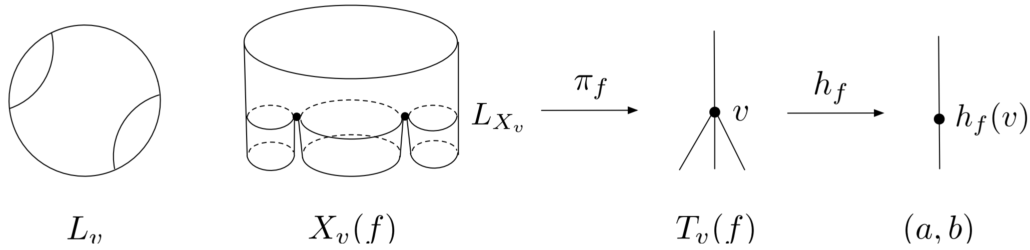

FIGURE 1.1. The local features of the quotient maps in (1.1) at a vertex v of $T(f)$ and the associated lamination L_v.

Local features. – See Figure 1.1. Every vertex v of $T(f)$ has a nongeneric height $h_f(v)$; and for each vertex, there is a maximal interval $(a,b) \subset (0,\infty)$ containing $h_f(v)$ for which all heights $t \in (a,b) - \{h_f(v)\}$ are generic. The connected component $X_v(f)$ of $G_f^{-1}(a,b)$ containing $\pi_f^{-1}(v)$ is a planar Riemann surface that we call a *local model surface*. The intersection $L_{X_v} := X_v(f) \cap \pi_f^{-1}(v)$ is called the *central leaf* of $X_v(f)$. The central leaf L_{X_v} is a connected component of a fiber of G_f, is homeomorphic to the underlying space of a finite planar graph, and is the boundary of the unbounded component of its complement. This implies it is naturally the quotient of the unit circle by a certain kind of equivalence relation, a *finite lamination*, denoted L_v. The lamination L_v is encoded by a simple picture in a disk, a *lamination diagram*. The lamination diagram is not endowed with coordinates—rotating the picture does not change it—but the circle is equipped with a metric induced by the 1-form $|\partial G_f|$. For example: if two critical points c_1, c_2 belong to the same central leaf L_{X_v}, their relative angles in L_v are determined by the metric.

The pictograph. – The pictograph $\mathcal{D}(f)$ is a diagram consisting of a collection of laminations L_v, associated to vertices in the convex hull of the critical points of $(F, T(f))$, together with labels that mark the orbits of the critical points. For illustration, Figure 1.2 shows a pictograph associated to a polynomial of degree 4 in the shift locus. The markings on each lamination indicate which iterate of a critical point lands on a central leaf or is seen through the "pant leg" of the local model surface. We emphasize two things.

1. The pictograph contains both combinatorial and metric information. For example, if the iterates of two critical points, say $f^i(c_1)$ and $f^j(c_2)$, both lie on a central leaf L_{X_v}, then the lamination L_v is labeled by symbols i_1 and j_2, placed on the unit circle at a distance recording the metric information of how these points are deployed in L_{X_v}.
2. The pictograph is a static object. It does not, by definition, include a self-map of an object. However, it allows for reconstruction of dynamics, as described in the main results presented below.

The next paragraph describes what the pictograph captures.

The tree of local models. – The *tree of local models* is the disjoint union $\mathcal{X}(f) := \bigsqcup_v X_v(f)$, indexed by the vertices v of the tree $T(f)$. It is equipped with a holomorphic self-map $\mathcal{F}$ induced by f. The collection of inclusions $\{X_v(f) \hookrightarrow X(f)\}_{v \in T(f)}$ induces a (generically two-to-one) *gluing quotient map* $g_f : \mathcal{X}(f) \to X(f)$, which is not part of the data of the tree of local models.