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HOCHSCHILD-PIRASHVILI HOMOLOGY ON SUSPENSIONS AND REPRESENTATIONS OF $\mathrm{Out}(F_n)$

BY VICTOR TURCHIN AND THOMAS WILLWACHER

ABSTRACT. – We show that the Hochschild-Pirashvili homology on any suspension admits the so called Hodge splitting. For a map between suspensions $f:\Sigma Y \to \Sigma Z$, the induced map in the Hochschild-Pirashvili homology preserves this splitting if f is a suspension. If f is not a suspension, we show that the splitting is preserved only as a filtration. As a special case, we obtain that the Hochschild-Pirashvili homology on wedges of circles produces new representations of $\mathrm{Out}(F_n)$ that do not factor in general through $\mathrm{GL}(n,\mathbb{Z})$. The obtained representations are naturally filtered in such a way that the action on the graded quotients does factor through $\mathrm{GL}(n,\mathbb{Z})$.

RÉSUMÉ. – On montre que l'homologie de Hochschild-Pirashvili sur toute suspension admet une certaine décomposition de Hodge. Pour toute application entre suspensions $f:\Sigma Y \to \Sigma Z$, l'application induite en homologie de Hochschild-Pirashvili préserve cette décomposition si f est une suspension. Dans le cas contraire, on montre que la décomposition est préservée uniquement en tant que filtration. Dans le cas particulier d'un bouquet de cercles, l'homologie de Hochschild-Pirashvili produit de nouvelles représentations de $\mathrm{Out}(F_n)$ qui ne se factorisent pas en général par $\mathrm{GL}(n,\mathbb{Z})$. Les représentations ainsi obtenues sont naturellement filtrées de façon à ce que l'action sur les quotients gradués se factorise par $\mathrm{GL}(n,\mathbb{Z})$.

0. Introduction

The higher Hochschild homology is a bifunctor introduced by T. Pirashvili in [29] that to a topological space (simplicial set) and a (co)commutative (co)algebra assigns a graded vector space. Informally speaking this functor is a way to "integrate" a (co)algebra over a given space. Specialized to a circle the result is the usual Hochschild homology. The precursor to the higher Hochschild homology was the discovery of the Hodge splitting in the usual Hochschild homology of a commutative algebra [13, 21]. Indeed, the most surprising and

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perhaps motivating result for T. Pirashvili to write his seminal work [29] was the striking fact that the higher Hochschild homology on a sphere of any positive dimension also admits the Hodge splitting and moreover the terms of the splitting up to a regrading depend only on the parity of the dimension of the sphere. With this excuse to be born, the higher Hochschild homology is nowadays a widely used tool that has various applications including the string topology and more generally the study of mapping and embedding spaces [29, 1, 2, 15, 26, 27, 32, 33]. It also has very interesting and deep generalizations such as the topological higher Hochschild homology [8, 31] and factorization homology [3, 14, 16, 23].

In our work we study the very nature of the Hodge splitting. In particular we show that it always takes place for suspensions. Moreover, it will be clear from the construction that only suspensions and spaces rationally homology equivalent to them have this property. For any suspension ΣY, the terms of the splitting depend in some polynomial way on $\tilde{H}_*\Sigma Y$, which in particular explains Pirashvili's result for spheres. We also show that if a map $f\colon \Sigma Y \to \Sigma Z$ is a suspension, then the induced map in the Hochschild-Pirashvili homology preserves the splitting and is determined by the map $f_*\colon \tilde{H}_*\Sigma Y \to \tilde{H}_*\Sigma Z$. In case f is not a suspension, the Hodge splitting is preserved only as a filtration. We explain how the induced map between different layers is computed from the rational homotopy type of f.

We treat more carefully the case of wedges of circles and discover certain representations of the group $\mathrm{Out}(F_n)$ of outer automorphisms of a free group [(1)] that have the smallest known dimension among those that don't factor through $\mathrm{GL}(n,\mathbb{Z})$.

Notation

We work over rational numbers $\mathbb{Q}$ unless otherwise stated. All vector spaces are assumed to be vector spaces over $\mathbb{Q}$. Graded vector spaces are vector spaces with a $\mathbb{Z}$-grading, and we abbreviate the phrase "differential graded" by dg as usual. We generally use homological conventions, i.e., the differentials will have degree -1. We denote by gVect and dgVect the category of graded vector spaces and the category of chain complexes respectively. For a chain complex or a graded vector space C we denote by $C[k]$ its k-th desuspension.

We use freely the language of operads. A good introduction into the subject can be found in the textbook [22], whose conventions we mostly follow. We use the notation $\mathcal{P}\{k\}$ for the k-fold operadic suspension. The operads governing commutative unital, associative unital, and Lie algebras are denoted by Com, Assoc, and Lie respectively. By Com_+ we denote the commutative non-unital operad and by coLie the cooperad dual to Lie.

For a small category $\mathcal{C}$, we denote by mod$-\mathcal{C}$ the category of cofunctors $\mathcal{C}^{\mathrm{op}} \to \mathrm{dgVect}$ to chain complexes. The objects of mod$-\mathcal{C}$ will be called *right $\mathcal{C}$-modules*. For right $\mathcal{C}$-modules X and Y, we denote by $\mathrm{Rmod}_{\mathcal{C}}(X,Y)$ the complex of natural transformations $X \to Y$, and by $\mathrm{hRmod}_{\mathcal{C}}(X,Y)$ the complex of homotopy natural transformations $X \to Y$ obtained as the right derived functor of $\mathrm{Rmod}_{\mathcal{C}}(-,-)$, see [37, Section 10.7]. In the following section, $\mathcal{C}$ is either the category Γ of finite pointed sets or the category Fin of

(1) These representations appear as application to the hairy graph-homology computations in the study of the spaces of long embeddings, higher dimensional string links, and the deformation theory of the little disks operads [2, 33, 34, 35, 36].

finite sets. Abusing notation we denote the set $\{1, \dots, k\}$ by k and the set $\{*, 1, \dots, k\}$ based at $*$ by k_*. We will consider the following examples of right Γ and Fin-modules:

- For X some topological space we can consider the Fin-module sending a finite set S to the singular chains on the mapping space $C_*(X^S)$. We denote this Fin-module by $C_*(X^\bullet)$.
- Similarly, to a basepointed space X_* we assign a Γ-module $C_*(X_*^\bullet)$ sending a pointed set S_* to $C_*(X_*^{S_*})$, where now $X_*^{S_*}$ is supposed to be the space of pointed maps.
- To a cocommutative coalgebra C we assign the Fin-module sending the finite set S to the tensor product $C^{\otimes S} \cong \bigotimes_{s \in S} C$. We denote this Fin-module by $C^{\otimes \bullet}$. If not otherwise stated we assume that C is non-negatively graded and simply connected (its degree zero and one parts are $\mathbb{Q}$ and 0, respectively).
- If in addition M is a C-comodule (e.g., $M = C$) one can construct a Γ-module $M \otimes C^{\otimes \bullet}$ such that $S_* \mapsto M \otimes \bigotimes_{s \in S_* \setminus \{*\}} C$.
- Dually, if M is a module over a commutative algebra A, then $M \otimes A^{\otimes \bullet}$ is a *left* Γ-module, and its objectwise dual $\left(M \otimes A^{\otimes \bullet}\right)^\vee$ is a right Γ-module.

A topological space is said of finite type if all its homology groups are finitely generated in every degree.

Two spaces are said rationaly homology equivalent if there is a zigzag of maps between them, such that its every map induces an isomorphism in rational homology.

The completed tensor product is denoted by $\hat{\otimes}$.

Main results

In the paper for simplicity of exposition we stick to the contravariant Hochschild-Pirashvili homology that is to the one assigned to right Fin and Γ modules. One should mention however that all the results can be easily adjusted to the covariant case as well.

There are two ways to define the higher Hochschild homology. In the first combinatorial way, for a space X (respectively pointed space X_*) obtained as a realization of a (pointed) finite simplicial set $\mathscr{X}_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Fin}$ (respectively $\mathscr{X}_\bullet \colon \Delta^{\mathrm{op}} \to \Gamma$), the higher Hochschild homology $HH^X(L)$ (respectively $HH^{X_*}(L_*)$) can be computed as the homology of the totalization of the cosimplicial chain complex $L \circ \mathscr{X} \colon \Delta \to \mathrm{dgVect}$ (respectively $L_* \circ \mathscr{X}_* \colon \Delta \to \mathrm{dgVect}$). [(2)]

In another definition, for a right Fin-module L (respectively right Γ-module L_*) and a topological space X (respectively pointed space X_*), the *higher Hochschild homology* that we also call *Hochschild-Pirashvili homology* $HH^X(L)$ (respectively $HH^{X_*}(L_*)$) is the homology of the complex of homotopy natural transformations $C_*(X^\bullet) \to L$ (respectively $C_*(X_*^\bullet) \to L_*$) [29, 16].

The fact that the two definitions are equivalent is implicitly shown in the proof of [29, Theorem 2.4] by Pirashvili, see also [16, Proof of Proposition 4] and [32, Proposition 3.4].

(2) This definition can also be adjusted to realizations of any simplicial sets non-necessarily finite by using the right Kan extention of L (respectively L_*) to the category of all (pointed) sets [29].