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ISING MODEL AND S-EMBEDDINGS OF PLANAR GRAPHS

BY DMITRY CHELKAK

ABSTRACT. – We discuss the notion of s-embeddings $\mathcal{S} = \mathcal{S}_{\mathcal{X}}$ of planar graphs carrying a nearest-neighbor Ising model. The construction of $\mathcal{S}_{\mathcal{X}}$ is based upon a choice of a global complex-valued solution $\mathcal{X}$ of the propagation equation for Kadanoff-Ceva fermions. Each choice of $\mathcal{X}$ provides an interpretation of all other fermionic observables as s-holomorphic functions on $\mathcal{S}_{\mathcal{X}}$. We set up a general framework for the analysis of such functions on s-embeddings $\mathcal{S}^\delta$ with $\delta \to 0$. Throughout this analysis, a key role is played by the functions $\mathcal{Q}^\delta$ associated with $\mathcal{S}^\delta$, the so-called origami maps in the bipartite dimer model terminology. In particular, we give an interpretation of the mean curvature of the limit of discrete surfaces $(\mathcal{S}^\delta; \mathcal{Q}^\delta)$ viewed in the Minkowski space $\mathbb{R}^{2,1}$ as the mass in the Dirac equation describing the continuous limit of the model.

We then focus on the simplest situation when $\mathcal{S}^\delta$ have uniformly bounded lengths/angles and $\mathcal{Q}^\delta = O(\delta)$; as a particular case this includes all critical Ising models on doubly periodic graphs via their canonical s-embeddings. In this setup we prove RSW-type crossing estimates for the random cluster representation of the model and the convergence of basic fermionic observables. The proof relies upon a new strategy and also provides a quantitative estimate on the speed of convergence.

RÉSUMÉ. – Nous introduisons la notion de s-plongements $\mathcal{S} = \mathcal{S}_{\mathcal{X}}$ de graphes planaires munis d'un modèle d'Ising. La construction de $\mathcal{S}_{\mathcal{X}}$ se base sur un choix d'une solution $\mathcal{X}$ à valeurs complexes de l'équation de propagation pour les fermions de Kadanoff-Ceva. Chaque choix de $\mathcal{X}$ donne une interprétation de toutes les autres observables fermioniques comme fonctions s-holomorphes sur $\mathcal{S}_{\mathcal{X}}$. Nous établissons un cadre général pour l'analyse de telles fonctions sur les s-plongements $\mathcal{S}^\delta$ avec $\delta \to 0$. Tout au long de cette analyse, un rôle essentiel est joué par les fonctions $\mathcal{Q}^\delta$ associées aux $\mathcal{S}^\delta$, dites applications origami dans la terminologie du modèle de dimères sur des graphes bipartis. En particulier nous donnons une interprétation de la courbure moyenne de la limite des surfaces discrètes $(\mathcal{S}^\delta; \mathcal{Q}^\delta)$ vue dans l'espace de Minkowski $\mathbb{R}^{2,1}$ comme la masse dans l'équation de Dirac décrivant la limite continue du modèle.

Par la suite nous nous concentrons sur la situation la plus simple où les $\mathcal{S}^\delta$ ont des longueurs/angles bornés uniformément et $\mathcal{Q}^\delta = O(\delta)$; cela comprend comme cas particulier tous les modèles d'Ising critiques sur des graphes doublement périodiques via leurs s-plongements canoniques. Dans ce cadre, nous démontrons des estimations de croisement de type RSW pour la représentation de Fortuin-Kasteleyn du modèle et la convergence des observables fermioniques les plus simples. La démonstration se base sur une nouvelle stratégie et fournit une estimation quantitative de la vitesse de convergence.

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1. Introduction, main results and perspectives

1.1. General context

The Ising model of a ferromagnet, introduced by Lenz in 1920, recently celebrated its centenary; probably, this is one of the most studied models in statistical mechanics. The *planar* Ising model (i.e., the 2D model with nearest-neighbor interactions) gives rise to surprisingly rich structures of correlation functions; we refer the reader to monographs [26, 44, 46, 48], lecture notes [24], as well as to the introductions of the papers [1, 18] and references therein for more background on the subject, discussed from a variety of perspectives.

In this paper we consider the planar Ising model without the magnetic field, which is known to be exactly solvable on any graph and at any temperature: the partition function can be written as the Pffafian of a certain matrix and the entries of the inverse matrix—known under the name *fermionic observables*—satisfy a simple *propagation equation*; we refer the reader to the paper [9] for more details on various combinatorial formalisms used to study the planar Ising model during its long history.

We prefer to work with the ferromagnetic Ising model defined on *faces* of a planar graph G; the partition function is given by

$$\mathcal{Z}(G) \;=\; \textstyle\sum_{\sigma:G^\circ\to\{\pm1\}} \exp\big[\,\beta \sum_{e\in E(G)} J_e\sigma_{v^\circ_-(e)}\sigma_{v^\circ_+(e)}\,\big], \tag{1.1}$$

where G° and $E(G)$ denote the dual graph and the set of edges of G, respectively; $\beta = 1/kT$ is the inverse temperature, $J_e > 0$ are interaction constants assigned to the edges of G, and $v^\circ_\pm(e) \in G^\circ$ denote the two faces adjacent to $e \in E(G)$. Passing to the *domain walls* representation (also known as the low-temperature expansion) of the model, one can rewrite $\mathcal{Z}(G)$ as

$$\mathcal{Z}(G) \;=\; 2\textstyle\prod_{e\in E(G)}(x(e))^{-1/2} \times \sum_{C\in\mathcal{E}(G)}\prod_{e\in C} x(e), \qquad x(e) := \exp[-2\beta J_e].$$

where $\mathcal{E}(G)$ denotes the set of all even subgraphs of G. Throughout this paper we identify edges of G with faces $z(e)$ of the graph $\Lambda(G) := G\cup G^\circ$ and denote

$$\theta_{z(e)} \;:=\; 2\arctan x(e) \;\in\; (0,\tfrac{1}{2}\pi), \tag{1.2}$$

Let us emphasize that we do *not* fix an embedding of G into $\mathbb{C}$ at this point, thus θ_e is nothing more than another (abstract, i.e., not geometric) *parametrization* of the interaction constants J_e.

We are mostly interested in the situation when (G,x) is a big weighted graph carrying critical or near-critical Ising weights $x(e)$ though we do *not* precise the exact sense of this condition and mention it here only to give a proper perspective. For instance, the reader can think about the setup in which G is a subgraph of an infinite doubly periodic graph, in this case the criticality condition on the collection of weights $x(e)$ is known explicitly [41, 19]. However, this periodic setup is *not* the main motivation for our paper and should be viewed as a very particular case though our results are new even in this situation and, in particular, answer a question posed by Duminil-Copin and Smirnov in their lecture notes on the conformal invariance of 2D lattice models; see [24, Question 8.5].

Given a weighted graph (G,x) we aim to embed it into the complex plane $\mathbb{C}$ (actually, we construct both an embedding of $\Lambda(G) := G\cup G^\circ$ and of its dual graph $\diamondsuit(G)$) in a way allowing to analyze (subsequential) limits of fermionic observables in the same spirit as in the seminal work of Smirnov [53, 54] on the critical Ising model on $\mathbb{Z}^2$. A possible analogy could be Tutte's barycentric embeddings which, among other things, provide a framework to study the convergence of discrete harmonic functions to continuous ones. Let us emphasize that *s-embeddings* introduced in our paper are *not* directly related to these barycentric embeddings; in fact both can be viewed as special cases of a more general construction called *t-embeddings* or *Coloumb gauges* appearing in the bipartite dimer model context [39, 15]. The notion of s-embeddings was first announced in [8] and motivated a closely related research project [15, 16]. We benefit from this interplay and partly rely upon results obtained in [15] in a more general context.

One of the most important (from our perspective) motivations to study special embeddings of irregular weighted planar graphs into $\mathbb{C}$ is the conjectural convergence of critical random maps carrying a lattice model to an object called *Liouville quantum gravity*; e.g., see [25, 27] or [51] and references therein. Though this discussion goes far beyond the scope of this paper, let us briefly mention several ideas of this kind that appeared in the literature: circle packing embeddings (see [47] and references therein), embeddings as piecewise flat Riemann surfaces (e.g., see [29] or [27, Section 1.7.1]), embeddings by square tilings for UST-decorated random maps, Tutte's embeddings [32, 31] and circle packings [30] for the mating-of-trees approach to the LQG, the so-called Cardy embeddings [33] introduced recently in the pure gravity context, etc. We believe that adding s-embeddings to this list might be a proper path for the analysis of random maps carrying the Ising model.

Besides generalizing the results of [54, 18] to a large family of deterministic graphs with 'flat' functions $\mathcal{Q}^\delta = O(\delta)$ and suggesting a general framework to attack problems coming from the random maps context, this paper also contains the following contributions to the subject:

— A new 'constructive' strategy of the proof of Theorem 1.2, which avoids compactness arguments in the Carathéodory topology on the set of planar domains and gives a quantitative estimate of the speed of convergence; see Section 4.3 for details. Due to a general framework recently developed in [52] this also allows to control the speed of convergence of FK-Ising interfaces to SLE(16/3) curves; see [52] for more details.

— In the non-flat case $\mathcal{Q}^\delta \not\to 0$ as $\delta \to 0$, we reveal the importance of embeddings of planar graphs carrying a nearest-neighbor Ising model into the *Minkowski space* $\mathbb{R}^{2,1}$ and not simply into the complex plane. As briefly discussed in Section 2.7, this leads to a rather unexpected interpretation of the *mass* appearing in the effective description of the model in the small mesh size limit as the *mean curvature* of the corresponding surface in $\mathbb{R}^{2,1}$.

The latter observation can be used in deterministic setups (e.g., see a recent paper [14] where such an interpretation of a near-critical model on isoradial grids is discussed) and leads to interesting questions on limits of random surfaces in $\mathbb{R}^{2,1}$ obtained via s-embeddings of appropriate random maps carrying the Ising model.