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THE REGULARITY OF ENVELOPES

BY ELEONORA DI NEZZA AND STEFANO TRAPANI

In memory of Jean-Pierre Demailly.

ABSTRACT. – Let X be a compact complex manifold of complex dimension n and α be a smooth closed real form on X such that its cohomology class $\{\alpha\} \in H^{1,1}(X, \mathbb{R})$ is big. In this paper we prove that, given a bounded function f with bounded distributional laplacian in X, the α-psh envelope $P_\alpha(f)$ is also locally bounded with locally bounded distributional laplacian on the ample locus of $\{\alpha\}$.

RÉSUMÉ. – Soit X une variété complexe compacte de dimension complexe n et α une (1,1)-forme réelle et fermée sur X telle que sa classe de cohomologie $\{\alpha\} \in H^{1,1}(X, \mathbb{R})$ soit grosse. Dans ce travail on démontre que, étant donnée une fonction bornée f dont le laplacien, au sens des distributions, est borné, alors l'enveloppe α-psh, $P_\alpha(f)$, est localement bornée et a aussi un laplacien borné sur le lieu ample de $\{\alpha\}$.

1. Introduction

Envelopes of plurisubharmonic functions played an important role in the development of the pluripotential theory on domains of $\mathbb{C}^n$, see for example [1, 2, 3, 27, 30].

When, relying on the Bedford and Taylor theory in the local case, the foundations of a pluripotential theory on compact Kähler manifolds have been developed [21, 22], *envelopes of quasi-plurisubharmonic functions* started to be intensively studied.

The geometric motivations we can mention are, among others, the study of geodesics in the space of Kähler metrics [10, 14, 6, 25, 12, 15, 11] and the transcendental holomorphic Morse inequalities on projective manifolds [29].

The two basic (and related) questions are about the regularity of envelopes and the behavior of their Monge-Ampère measures. To fix the setting, let X be a compact complex manifold of complex dimension n, let α be a smooth closed real $(1,1)$-form and let f be a

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function on X bounded from above. We are going to refer to f as "barrier function". Then the prototype of an envelope construction is

$$P_\alpha(f) := (\sup\{\varphi \in \mathrm{PSH}(X,\alpha) \,:\, \varphi \le f\})^*,$$

where * denotes the upper semi-continuous regularization and $\mathrm{PSH}(X,\alpha)$ is the space of all α-plurisubharmonic functions, as defined in Section 2.

The function $P_\alpha(f)$ is either a genuine α-plurisubharmonic function or identically $-\infty$. When $f = -\mathbf{1}_T$ is the negative characteristic function of a subset T, then $P_\alpha(f) = f_T^*$ is the so-called relative extremal function of T [21]. When $f = 0$, then $P_\alpha(0) = V_\alpha$ is a distinguished potential with minimal singularities.

We recall that a function on some open set in X is said to be in $C^{1,\bar{1}}$ if it is locally L^∞ and its distributional Laplacian is represented by a locally bounded function. Any $C^{1,\bar{1}}$ function is also $C^{1,\alpha}$ for any $0 < \alpha < 1$. In this paper we prove the following:

THEOREM 1.1. – *Let α be a real closed $(1,1)$-form such that $\{\alpha\}$ is a big class. Assume $f \in C^{1,\bar{1}}(X)$. Then $P_\alpha(f)$ is $C^{1,\bar{1}}$ on the ample locus of $\{\alpha\}$.*

We actually prove a regularity result for the more general rooftop envelope $P_\alpha(f_1,\dots,f_k)$ (see Theorem 4.2).

The study of such envelopes has led to several works. We start by summarizing them in the case of a smooth barrier function f.

The first result to mention is [5], where the author proves that in the case $\alpha \in c_1(L)$ where L is a big line bundle over X, the envelope $P_\alpha(f)$ is $C^{1,1}$ on the ample locus $\mathrm{Amp}(\{\alpha\})$ of α, and moreover

$$\alpha^n_{P_\alpha(f)} = \mathbf{1}_{\{P_\alpha(f)=f\}}\alpha_f^n. \tag{1.1}$$

Here, given a α-psh function u, $\alpha_u := \alpha + i\partial\bar{\partial}u$ and α_u^n denotes the non-pluripolar Monge-Ampère measure $(\alpha + i\partial\bar{\partial}u)^n$ (see [9, Definition 1.1]). Thus the left-hand side of (1.1) is the non-pluripolar product of $P_\alpha(f)$ while α_f^n is the wedge product n times of the form α_f that has bounded coefficients since f is smooth, (observe that f is not α-plurisubharmonic so the non-pluripolar product of α_f does not make sense).

Later people started to work on possible generalizations of the above results in the case of a pseudoeffective class $\{\alpha\}$ that does not necessarily represent the first Chern class of a line bundle. Assuming that $\{\alpha\}$ is big and nef, Berman [7], using PDE methods, proved that the envelope $P_\alpha(f)$ is $C^{1,\bar{1}}$ on $\mathrm{Amp}(\{\alpha\})$ and that the identity in (1.1) holds.

The optimal regularity $C^{1,1}$ when $\{\alpha\}$ is a Kähler class was then proved independently by [28] and [13], while the big and nef case was settled in [12].

For a general pseudoeffective class the equality

$$\alpha^n_{P_\alpha(f)} = \mathbf{1}_{\{P_\alpha(f)=f\}}\alpha_f^n \tag{1.2}$$

is established in [20], without relying on the regularity of $P_\alpha(f)$.

The general case of a big class was treated for the first time by Berman and Demailly. In [4] the authors claim to prove the $C^{1,\bar{1}}$ regularity of $P_\alpha(f)$ on the ample locus of $\{\alpha\}$ when f is smooth. However it became clear later that their arguments had a mistake in their crucial technical Lemma 1.12. In fact, the lower bound in their Lemma 1.12 does not follow from [4, eq. (1.8)] (as they state) since there are some mixed terms to take care of.

The purpose of this paper is then to correct the proof in [4] and generalize their regularity result.

Our proof relies on results in [4] and especially on [17] but it presents two novelties. We proceed in three steps:

- As in [4], we consider the regularization of the envelope through the exponential map. We first derive weaker estimates than the one stated in [4, Lemma 1.12], which allows us to obtain Hölder regularity of the envelope. This is done in Lemma 3.1.
- Such regularity and a version of the Lelong-Jensen inequality, allow us to prove an estimate for the complex hessian of the regularized function (like the one in Lemma 1.12) but with variable coefficients (Lemma 3.5). We can however control the asymptotics of these coefficients near the boundary of the ample locus of $\{\alpha\}$.
- In the third step, we pass to a modification, and we transport the regularized function on a suitable line bundle. Thanks to a change of coordinates we are able to "kill" the coefficients appearing in the lower bound of the hessian, thus reducing to estimates with constant coefficients. This is the key to use (more or less) the same strategy as in [4] and complete the proof. This is done in Section 4.

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2. Preliminaries

Let us first assume that (X, ω) is a compact Kähler manifold of complex dimension $n \geq 1$. The Kähler assumption will be then removed at the end of Section 4.

Let α be a smooth closed real $(1, 1)$ form such that the cohomology class $\{\alpha\}$ is big.

The cohomology class $\{\alpha\}$ is said to be big if there exists a positive closed $(1, 1)$-current $T = \alpha + i\partial\bar{\partial}\Phi \in \{\alpha\}$ such that $T \geq \varepsilon_0\omega$, for some $\varepsilon_0 > 0$. By replacing ω with $\varepsilon_0\omega$ we can assume $T \geq \omega$. Such currents are called Kähler currents.

Let Ω be the set of x in X such that there exists a Kähler current cohomologous to α which is smooth in a neighborhood of x. This is called the ample locus of $\{\alpha\}$ (also denoted by $\mathrm{Amp}\{\alpha\}$ in the literature). If $\{\alpha\}$ is big, Ω is a non-empty open subset such that $\Sigma = X \setminus \Omega$ is a closed analytic set. Moreover, there exists a Kähler current $T = \alpha + i\partial\bar{\partial}\Phi$ with analytic singularities which is smooth in Ω (see [8]). Observe that, without loss of generality, we can normalize $\Phi \leq -1$.

The cohomology class of α is Kähler if and only if $\Omega = X$. Also, Φ is a smooth function and we can take $\omega = \alpha + i\partial\bar{\partial}\Phi$.

A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is called quasi-plurisubharmonic if locally $\varphi = \rho + u$, where ρ is smooth and u is a plurisubharmonic function. We say that φ is α-plurisubharmonic