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STRONG PROPERTY (T), WEAK AMENABILITY AND ℓ^p-COHOMOLOGY IN $\tilde{A}_2$-BUILDINGS

BY Jean LÉCUREUX, Mikael DE LA SALLE
AND Stefan WITZEL

ABSTRACT. – We prove that cocompact (and more generally: undistorted) lattices on $\tilde{A}_2$-buildings satisfy Lafforgue's strong property (T), thus exhibiting the first examples that are not related to algebraic groups over local fields. Our methods also give two further results. First, we show that the first ℓ^p-cohomology of an $\tilde{A}_2$-building vanishes for any finite p. Second, we show that the non-commutative L^p-space for p not in $[\frac{4}{3}, 4]$ and the reduced C^*-algebra associated to an $\tilde{A}_2$-lattice do not have the operator space approximation property and, consequently, that the lattice is not weakly amenable.

RÉSUMÉ. – Nous montrons que les réseaux cocompacts (et plus généralement non distordus) dans les immeubles de type $\tilde{A}_2$ ont la propriété (T) renforcée de Lafforgue, ce qui fournit les premiers exemples qui ne proviennent pas de groupes algébriques sur des corps locaux. Les mêmes méthodes permettent d'obtenir d'autres résultats. D'une part nous montrons que, pour tout p fini, la cohomologie ℓ^p d'un immeuble de type $\tilde{A}_2$ s'annule en degré 1. D'autre part, nous montrons que l'espace L^p non commutatif pour p en dehors de l'intervalle $[\frac{4}{3}, 4]$ ainsi que la C^*-algèbre réduite associés à un réseau dans immeuble de type $\tilde{A}_2$ n'ont pas la propriété d'approximation au sens des espaces d'opérateurs. Par conséquent, un tel réseau n'est pas faiblement moyennable.

Introduction

Property (T) was introduced by Kazhdan in order to study algebraic properties of lattices in simple Lie groups. It turned out to be a powerful tool in many areas of mathematics, from geometric group theory to ergodic theory or operator algebras. In [29], V. Lafforgue, with K-theoretic applications in mind, introduced a property called *strong property (T)*. This property can be formulated as an extension of property (T) for representations on Hilbert spaces which are not necessarily unitary, but with a mild growth condition on the

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norm. Lafforgue also considered variants for representations on (some) Banach spaces, see Definition 2.1 for a precise definition. Such non-unitary representations indeed appear in Lafforgue's work on the Baum-Connes conjecture, so strong property (T) is a natural obstruction for this approach to work for some higher-rank lattices, see [31]. Further motivation for studying them include the study of actions on hyperbolic graphs [29], on Banach spaces [30], on manifolds [11, 10, 9], or the study of the geometry of Banach spaces and expander graphs [29, 30].

The main result of [29] is that for a local field F, Archimedean or not, the group $\mathrm{SL}_3(F)$, and more generally every almost simple algebraic group over F whose Lie algebra contains the Lie algebra of $\mathrm{SL}_3(F)$, as well as their cocompact lattices have strong property (T). For the Banach space variants, the results from [29] were, in the non-Archimedean case, improved in [30] to cover all Banach spaces with nontrivial Rademacher type, see § 3. Such results are not known for Lie groups, despite some efforts [54, 26]. Lafforgue's results were extended in [33, 27] to all higher rank simple algebraic groups, and later to their non-cocompact lattices [56]. Strong property (T) played an important role in the recent proof of Zimmer's conjecture [11, 10, 9]. However, the only known examples so far are related to (lattices in) simple algebraic groups.

We prove strong property (T) for $\tilde{A}_2$-lattices, providing the first examples outside the realm of algebraic groups:

MAIN THEOREM. – *Let X be an $\tilde{A}_2$-building, and let Γ be an undistorted lattice on X. Then Γ has strong property (T) with respect to every admissible Banach space. The class of admissible Banach spaces contains L^p spaces and more generally subspaces of quotients of non-commutative L^p spaces (*$1 < p < \infty$).

Recall that if F is a non-Archimedean local field then $\mathrm{SL}_{n+1}(F)$ acts on a Bruhat-Tits $\tilde{A}_n$-building [12, 13]. Conversely every $\tilde{A}_n$-building with $n \geq 3$ arises this way (possibly with F non-commutative) [58, 63]. However, there are infinitely many isomorphism classes of $\tilde{A}_2$-buildings that are not Bruhat-Tits and that admit lattices (called $\tilde{A}_2$-*lattices*) [3, Section 10], [51, Corollary E]. While arithmetic lattices in $\mathrm{SL}_n(F)$ are undistorted [35], we need to assume undistortedness in our result. Note, however, that all known non-arithmetic $\tilde{A}_2$-lattices are cocompact and therefore undistorted.

These non-arithmetic $\tilde{A}_2$ lattices, while being geometrically analogous to lattices in $\mathrm{SL}_3(F)$, exhibit some algebraic properties which are quite different from them, and in fact, from any linear group. For example, it has been proven that any linear representation of a non-arithmetic, cocompact $\tilde{A}_2$-lattice has finite image [3]. It is moreover conjectured that these groups are virtually simple.

The class of admissible Banach spaces is introduced in Section 5.4 and formally depends on the building. In particular, the complete statement about which Banach spaces we show to be admissible is Theorem 5.11. We conjecture a Banach space is admissible if and only if it has nontrivial type (Conjecture 5.13), as it is the case for Bruhat-Tits buildings.

Strong property (T) immediately translates in terms of fixed points for affine actions [30, Proposition 5.6]:

COROLLARY A. – *Let Γ be an undistorted $\tilde{A}_2$-lattice. Then for every admissible Banach space E every action of Γ by isometries on E has a fixed point.*

In particular, we obtain the following generalization of [41], which deals only with sufficiently small $p > 2$ (the precise condition depends on the building, but always implies $p \leq 2.106$) and small bounds on the uniformly bounded representation.

COROLLARY B. – *Let Γ be an undistorted $\tilde{A}_2$-lattice. For every uniformly bounded representation π of Γ on a L^p space ($1 < p < \infty$), or more generally a subspace of a quotient of a non-commutative L^p space ($1 < p < \infty$), one has $H^1(\Gamma, \pi) = 0$.*

Here a *non-commutative L^p space* is the L^p-space of a von Neumann algebra [49], see also § 3.4.

This is in sharp contrast with groups acting properly and cocompactly on bounded degree hyperbolic graphs, which admit proper actions on L^p spaces for $p < \infty$ large enough [65], and proper actions on quotients of L^p spaces for $p > 1$ small enough [2].

Kazhdan's property (T) was already known to hold for cocompact $\tilde{A}_2$-lattices by the work of Cartwright and Młotkowski [17] and by Zuk [66] and Pansu [44] (this is generalized to non cocompact lattices in Theorem 9.3). The proof of property (T) by Żuk and Pansu relies on a spectral estimate of links on vertices (see also the book [5] for a nice exposition). Our proof of strong property (T) is quite different and follows more closely Lafforgue's strategy for lattices in $\mathrm{SL}_3(F)$. Indeed, strong property (T) deals with representations with exponential growth rate at most some small number α, that is, satisfying

$$\exists C, \forall g, \|\pi(g)\| \leq Ce^{\alpha|g|}.$$

Therefore a proof of strong property (T) has to involve a local analysis at infinitely many different scales: local because the representation is a priori unbounded on the group, and at different scales because a representation with small exponential growth rate α but large constant C cannot be distinguished from a representation with large exponential growth rate.

Lafforgue's strategy can be outlined as follows (see also §2.2 below): first, he induces the representation (with small exponential growth rate) of Γ to the ambient algebraic group $G = \mathrm{SL}_3(F)$. Then he defines a sequence of suitably chosen averaging operators on G, and is able to obtain the desired convergence by a clever use of harmonic analysis on a maximal compact subgroup K of G. The first difference in our proof is that there is no ambient group G to work with. We are therefore led to do a more "geometric" induction, to a space of functions on the building. The main difficulty is then to replace the harmonic analysis on K by a careful study of the (asymptotic) geometry of balls in the building.

It turns out that the tools used in the proof of the main theorem allow us to derive some other interesting properties of $\tilde{A}_2$-buildings and their lattices. The first is a vanishing result for ℓ^p-cohomology of $\tilde{A}_2$-buildings. Recall that ℓ^p cohomology is a quasi-isometry invariant popularized by Gromov [20].

THEOREM C. – *Let X be a locally finite $\tilde{A}_2$-building. Then $\ell^p H^1(X) = 0$ for every $1 < p < \infty$.*