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ON THE PURITY CONJECTURE OF NISNEVICH FOR TORSORS UNDER REDUCTIVE GROUP SCHEMES

BY ROMAN FEDOROV

ABSTRACT. – Let R be a regular semilocal integral domain containing an infinite field k. Let $f \in R$ be an element such that for all maximal ideals $\mathfrak{m}$ of R we have $f \notin \mathfrak{m}^2$. Let $\mathbf{G}$ be a reductive group scheme over R. Under an isotropy assumption on $\mathbf{G}$ we show that a $\mathbf{G}$-torsor over the localization R_f is trivial, provided that it is rationally trivial. We show that this is not true without the isotropy assumption. Finally, if $\mathbf{G}$ is a commutative group scheme of multiplicative type and the regular semilocal ring contains a field of characteristic zero, we prove an analogue of Nisnevich purity conjecture for higher étale cohomology groups.

The first statement is derived from its abstract version concerning presheaves of pointed sets satisfying some properties. The counterexample is constructed by providing a torsor over a local family of affine lines that cannot be extended to a torsor over the corresponding local family of projective lines. The latter is accomplished using the technique of affine Grassmannians.

RÉSUMÉ. – Soit R anneau intègre semi-local régulier contenant un corps k infini. Soit $f \in R$ un élément tel que pour tous les idéaux maximaux $\mathfrak{m}$ de R on a $f \notin \mathfrak{m}^2$. Soit $\mathbf{G}$ un schéma en groupes réductifs sur R. Sous une hypothèse d'isotropie sur $\mathbf{G}$, nous montrons qu'un $\mathbf{G}$-torseur sur le localisé R_f est trivial, à condition qu'il soit rationnellement trivial. Nous montrons que cette affirmation ne tient pas sans l'hypothèse d'isotropie. Enfin, si $\mathbf{G}$ est un schéma en groupes commutatifs de type multiplicatif et l'anneau semi-local régulier contient un corps de caractéristique zéro, nous prouvons un analogue de la conjecture de pureté de Nisnevich pour les cohomologies étales supérieures.

La première affirmation est dérivée de sa version abstraite concernant les préfaisceaux d'ensembles pointés satisfaisant certaines propriétés. Le contre-exemple est obtenu en construisant un torseur sur une famille locale de droites affines qui ne s'étend pas en un torseur au-dessus de la famille correspondante de droites projectives. Cette dernière est accomplie à l'aide de la technique des grassmanniennes affines.

1. Introduction and main results

Let R be a regular local ring, let $\mathbf{G}$ be a reductive group scheme over R, and let $\mathcal{E}$ be a $\mathbf{G}$-torsor over R. The famous Grothendieck-Serre conjecture predicts that $\mathcal{E}$ is trivial, if it

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is trivial over the fraction field K of R. This has been proved in the case when R contains a field [13, 25]. In fact, a generalization of this conjecture to *semilocal* regular integral domains containing fields has been proved. The conjecture is still wide open in the mixed characteristic case, though recently there has been some significant progress [12, 4, 20, 19].

The conjecture [24, Conj. 1.3], that was formulated by Nisnevich in 1989, is a generalization of the Grothendieck-Serre conjecture: *let R and $\mathbf{G}$ be as before, let $f \in R$ be such that $f \notin \mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of R, and let $\mathcal{E}$ be a $\mathbf{G}$-torsor over R_f, then $\mathcal{E}$ is trivial, if it is trivial over K.* This conjecture reduces to the Grothendieck-Serre conjecture if $f = 1$. Very little was known about this conjecture until now. If R is a discrete valuation ring, the conjecture reduces to the Grothendieck-Serre conjecture if f is invertible and is trivial otherwise. Nisnevich in [24] proved the conjecture in dimension two, assuming that the residue field of the local ring R is infinite and that $\mathbf{G}$ is quasi-split (the condition that the residue field is infinite is removed by Česnavičius, see [5, Sect. 3.4.2(3)]). Note that the Nisnevich conjecture is non-trivial even for $\mathbf{G} = \mathrm{GL}_n$, as opposed to the Grothendieck-Serre conjecture (in the GL_n case the conjecture goes back to Quillen, see [30]). In this case the conjecture was proved by Bhatwadekar and Rao in [2], assuming that R is a localization of a finitely generated algebra over a field. This was extended to arbitrary regular local rings containing fields by Popescu [29, Thm. 1]. Guo proved a version of this conjecture for the rings of formal power series over valuation rings [18, Cor. 7.5].

In this article, if the regular local ring contains an infinite field, we prove the conjecture under some isotropy condition on $\mathbf{G}$. In fact, under these assumptions, we prove the generalization of the conjecture to semilocal regular integral domains. Then we show that the conjecture does not hold without the isotropy assumption.

Finally, if $\mathbf{G}$ is a commutative group scheme of multiplicative type and the regular semilocal ring contains a field of characteristic zero, we prove an analogue of Nisnevich purity conjecture for higher étale cohomology groups.

1.1. Notations and conventions

If C is a category, we denote by $\mathrm{Ob}(C)$ its class of objects.

If A is a pointed set, we denote its distinguished element by $\star$.

If X is an integral scheme, we denote by $K(X)$ its function field. If x is a schematic point of X, we denote by $\mathcal{O}_{X,x}$ its local ring and by $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ the corresponding maximal ideal. More generally, if $\mathbf{x} \subset X$ is a set of schematic points contained in an open affine subset of X, we denote by $\mathcal{O}_{X,\mathbf{x}}$ the corresponding semilocal ring.

We often work over a fixed base field k, in that case the product is the product over k: $\times := \times_k$.

Let U be a scheme and $\mathbf{G}$ be a flat U-group scheme finitely presented over U. A U-scheme $\mathcal{E}$ with a left action $\mathrm{act}\colon \mathbf{G} \times \mathcal{E} \to \mathcal{E}$ is called *a $\mathbf{G}$-torsor over U* if $\mathcal{E}$ is faithfully flat and quasi-compact over U and the action is simply transitive, that is, the morphism $(\mathrm{act}, p_2)\colon \mathbf{G} \times_U \mathcal{E} \to \mathcal{E} \times_U \mathcal{E}$ is an isomorphism (see [17, Sect. 6]). A $\mathbf{G}$-torsor $\mathcal{E}$ over U is *trivial* if $\mathcal{E}$ is isomorphic to $\mathbf{G}$ as a U-scheme with an action of $\mathbf{G}$. This is well-known to be equivalent to the projection $\mathcal{E} \to U$ having a section. If T is a U-scheme, we will use the term "$\mathbf{G}$-torsor over T" to mean a "$\mathbf{G}_T$-torsor over T".

Assume that $\mathbf{G}$ is affine over U. The pointed set of isomorphism classes of $\mathbf{G}$-torsors over T is the first non-abelian cohomology group $H^1(T, \mathbf{G}) := H^1_{\mathrm{fppf}}(T, \mathbf{G})$ (note that, since $\mathbf{G}$ is affine, every class in $H^1_{\mathrm{fppf}}(T, \mathbf{G})$ gives rise to a $\mathbf{G}$-torsor because affine schemes can be glued in the fppf topology). The $\mathbf{G}$-torsors are also called principal $\mathbf{G}$-bundles.

1.2. Nisnevich purity conjecture

Let us recall a notion from [11, Def. 1.1]. Let $\mathbf{G}$ be a semisimple group scheme of adjoint type over a connected scheme U (see [1, Exp. XXII, Def. 4.3.3]). By [1, Exp. XXIV, Prop. 5.10] there is a sequence $U_1, \ldots, U_r$ of finite étale connected U-schemes such that

$$\mathbf{G} \simeq \prod_{i=1}^{r} \mathbf{G}^i, \tag{1}$$

where $\mathbf{G}^i$ is the Weil restriction of a simple adjoint U_i-group scheme (see [1, Exp. XXIV, Sect. 5.3] for the definition of a simple group scheme). Note that the group schemes $\mathbf{G}^i$ are uniquely defined by $\mathbf{G}$ up to isomorphism. Recall that a semisimple group scheme $\mathbf{G}$ over a scheme S is called *isotropic*, if it contains a non-trivial split torus. If S is connected and semilocal, then a semisimple group scheme $\mathbf{G}$ is isotropic if and only if it contains a proper parabolic subgroup scheme (see [1, Exp. XXVI, Cor. 6.14]).

Definition 1.1. – We say that a semisimple U-group scheme $\mathbf{G}$ of adjoint type is *strongly locally isotropic* if each factor $\mathbf{G}^i$ of $\mathbf{G}$ is isotropic Zariski locally over U.

We send the reader to [11, Rem. 1.2] for the discussion of this notion.

Let now $\mathbf{G}$ be a reductive U-group scheme. Let $\mathbf{Z}$ be the center of $\mathbf{G}$. We have the adjoint group scheme $\mathbf{G}^{\mathrm{ad}} := \mathbf{G}/\mathbf{Z}$ (see [1, Exp. XXII, Prop. 4.3.4(i), Prop. 4.3.5(ii), Def. 4.3.6]).

We prove the Nisnevich conjecture assuming that the regular semilocal integral domain contains an infinite field and the adjoint group scheme of $\mathbf{G}$ is strongly locally isotropic.

Theorem A. – *Assume that R is a regular semilocal integral domain containing an infinite field k. Let $K = K(R)$ be the quotient field of R. Let $f \in R$ be such that $f \notin \mathfrak{m}^2$ for any maximal ideal $\mathfrak{m}$ of R. Let $\mathbf{G}$ be a reductive R-group scheme such that $\mathbf{G}^{\mathrm{ad}}$ is strongly locally isotropic. Let $\mathcal{E}$ be a $\mathbf{G}$-torsor over R_f such that its pullback to K is trivial. Then $\mathcal{E}$ is trivial. In other words, the natural map $H^1(R_f, \mathbf{G}) \to H^1(K, \mathbf{G})$ has a trivial kernel.*

Remark 1.2. – We note that our proof was extended by Česnavičius in [6] to the case when k is allowed to be finite. It is also shown in loc. cit. that it is enough to assume that $\mathbf{G}$ is strongly locally isotropic along the hypersurface $\{f = 0\}$.

We will axiomatize the properties of the first non-abelian cohomology functor needed to prove the statement (see Theorem E below). We derive the above theorem from its "abstract" version in Section 3.2.

The Nisnevich conjecture is not true in general. We indeed provide counterexamples when $\mathbf{G}^{\mathrm{ad}}$ fails to be strongly locally isotropic.

Theorem B. – *There are a regular local ring R containing an infinite field, a simple simply-connected group scheme $\mathbf{G}$ over R, and a generically trivial $\mathbf{G}$-torsor $\mathcal{E}$ over $\mathbb{A}^1_R$ that cannot be extended to $\mathbb{P}^1_R$.*