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COUNTING CUSP FORMS BY ANALYTIC CONDUCTOR

BY Farrell BRUMLEY AND Djordje MILIĆEVIĆ

ABSTRACT. – Let F be a number field and $n \geqslant 1$ an integer. The *universal family* is the set $\mathfrak{F}$ of all unitary cuspidal automorphic representations on GL_n over F, ordered by their analytic conductor. We prove an asymptotic for the size of the truncated universal family $\mathfrak{F}(Q)$ as $Q \to \infty$, under a spherical assumption at the archimedean places when $n \geqslant 3$. We interpret the leading term constant geometrically and conjecturally determine the underlying Sato-Tate measure. Our methods naturally provide uniform Weyl laws with logarithmic savings in the level and strong quantitative bounds on the non-tempered discrete spectrum for GL_n.

RÉSUMÉ. – Soient F un corps de nombres et $n \geqslant 1$ un entier. La *famille universelle* $\mathfrak{F}$ est l'ensemble de toutes les représentations cuspidales unitaires automorphes de GL_n sur F, muni de l'ordre induit par le conducteur analytique. Nous obtenons un équivalent asymptotique pour le cardinal de la famille universelle tronquée $\mathfrak{F}(Q)$ lorsque $Q \to \infty$, sous une hypothèse de sphéricité aux places archimédiennes si $n \geqslant 3$. Nous interprétons géométriquement le terme dominant and déterminons conjecturalement la mesure de Sato-Tate sous-jacente. Nos méthodes fournissent une loi de Weyl uniforme avec un gain logarithmique dans le niveau et des bornes quantitatives fortes sur le specter discret non tempéré pour GL_n.

While automorphic forms can be notoriously difficult to study individually using analytic techniques, desired results can often be obtained by embedding them into a larger family of cusp forms of favorable size. In this article, we address the question of the asymptotic size of the universal family, which contains all cuspidal automorphic representations on GL_n over a fixed number field F, and is ordered by the analytic conductor of Iwaniec and Sarnak [40].

The analytic conductor $Q(\pi)$ of a cusp form π is an archimedean fattening of the classical arithmetic conductor of Casselman [15] and Jacquet-Piatetski-Shapiro-Shalika [43]:

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indeed, it is the product of the conductors $q(\pi_v)$ of all local components π_v, each of these arising from the local functional equation of the standard L-function $L(s, \pi_v)$. One way to understand the significance of the analytic conductor to the theory of L-functions is that its square-root determines the effective length of the partial sums appearing in the global approximate functional equation for $L(s, \pi)$. In turn, the analytic conductor controls the complexity of an array of analytic problems involving L-functions, such as the evaluation of moments, subconvexity, nonvanishing, extreme value problems, and numerical computations. From a broader and related perspective, $Q(\pi)$ quantifies the size of a system of equations large enough to pin down π exactly. In this respect, it has a close connection with the requisite number of twists in the Converse Theorem.

Our interest here is in the role that $Q(\pi)$ plays as a natural height function in the automorphic context. To this end, we denote by $\mathfrak{F}$ the countable discrete set of all irreducible unitary cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_F)$, considered up to unitary twist $|\det|^{it}$, organized into a family under the ordering induced by $Q(\pi)$. Following [75], we shall refer to $\mathfrak{F}$ as the *universal family*. In recent years, Sarnak has repeatedly emphasized the importance of understanding the statistical properties of the set

$$\mathfrak{F}(Q) = \{\pi \in \mathfrak{F} : Q(\pi) \leqslant Q\}.$$

It may come as a surprise how little is known about $\mathfrak{F}(Q)$.

In this paper we investigate the cardinality $|\mathfrak{F}(Q)|$, for increasing Q. Historically, the first result in this direction is the finiteness of $\mathfrak{F}(Q)$, which was established in [12] (see also [60]). Later, Michel and Venkatesh [57] showed that $|\mathfrak{F}(Q)|$ has at most polynomial growth in Q.

Our main theorem is the determination of the asymptotic size of $\mathfrak{F}(Q)$, subject to a spherical hypothesis on the archimedean component of π when $n > 2$. This allows us to answer in the affirmative the question posed by Michel-Venkatesh in [57] regarding the limiting behavior of $\log|\mathfrak{F}(Q)|/\log Q$. More precisely, we find that $|\mathfrak{F}(Q)|$ has pure power growth of the order Q^{n+1}, with no logarithmic factors. Moreover, we interpret the leading term constant in a way which is consistent with analogous problems for counting rational points of bounded height.

1. Introduction

Having briefly described the central problem, we now proceed to stating more precisely our main asymptotic result on the counting function of $\mathfrak{F}(Q)$, the trace formula input on which it depends, as well as an interpretation of the leading term constant.

1.1. Weyl-Schanuel law

We formulate, in Conjecture 1 below, the expected asymptotic behavior of $|\mathfrak{F}(Q)|$. Following [77], we refer to this asymptotic as the *Weyl-Schanuel law*. Indeed, it can simultaneously be viewed as a sort of universal Weyl law, and as an automorphic analogue to Schanuel's well-known result on the number of rational points of bounded height on projective spaces.

To describe the conjecture, we shall need to set up some notation. Let $\Pi(\mathrm{GL}_n(\mathbb{A}_F))$ denote the restricted direct product, over all places v of F, of the local unitary duals of

$\mathrm{GL}_n(F_v)$ relative to the local unramified duals. Let $\Pi(\mathrm{GL}_n(\mathbb{A}_F)^1)$ be the subset consisting of those π whose central characters are trivial on the diagonal embedding of the positive reals. We give $\Pi(\mathrm{GL}_n(\mathbb{A}_F)^1)$ the subspace topology derived from the direct product topology. We may embed the universal family $\mathfrak{F}$ into $\Pi(\mathrm{GL}_n(\mathbb{A}_F)^1)$ by taking local components, and the notion of analytic conductor extends to all of the latter space.

Let $\mathrm{GL}_n(\mathbb{A}_F)^1$ be the subgroup of $g \in \mathrm{GL}_n(\mathbb{A}_F)$ with $|\det g|_{\mathbb{A}_F} = 1$. We equip $\mathrm{GL}_n(\mathbb{A}_F)^1$ with Tamagawa measure, denoted by ω_{GL_n}; in particular, ω_{GL_n} assigns the automorphic quotient $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)^1$ volume 1. This then induces a normalization of Plancherel measure $\widehat{\omega}^{\mathrm{pl}}$ on $\Pi(\mathrm{GL}_n(\mathbb{A}_F)^1)$. Let

$$\mathscr{C}(\mathfrak{F}) = \frac{1}{n+1} \int_{\Pi(\mathrm{GL}_n(\mathbb{A}_F)^1)} Q(\pi)^{-n-1}\, \mathrm{d}\widehat{\omega}^{\mathrm{pl}}(\pi), \tag{1.1}$$

with the integral being regularized as in §1.4.

We may now state the following

CONJECTURE 1 (Weyl-Schanuel law). – *Fix a number field F and an integer $n \geqslant 1$. Then*

$$|\mathfrak{F}(Q)| \sim \mathscr{C}(\mathfrak{F})Q^{n+1} \quad \textit{as } Q \to \infty.$$

Conjecture 1 is motivated by the heuristic, borrowed from the setting of Schanuel's theorem, that the asymptotic behavior of $|\mathfrak{F}(Q)|$ should align with that of the "Plancherel volume of the conductor ball":

$$V_{\mathfrak{F}}(Q) = \int_{\substack{\pi\in\Pi(\mathrm{GL}_n(\mathbb{A}_F)^1)\\ Q(\pi)\leqslant Q}} \mathrm{d}\widehat{\omega}^{\mathrm{pl}}(\pi). \tag{1.2}$$

In Proposition 6.1 an asymptotic evaluation of the finite integral $V_{\mathfrak{F}}(Q)$ is given, in which $\mathscr{C}(\mathfrak{F})$ appears as the leading term constant. Note that $V_{\mathfrak{F}}(Q)$ contains no automorphic information.

In this paper, we establish the above predicted asymptotics for $|\mathfrak{F}(Q)|$ in many cases, with explicit logarithmic savings in the error term. Namely, we prove the following

THEOREM 1.1. – *Let F be a number field and $n \geqslant 1$ an integer. The Weyl-Schanuel law holds for $n \leqslant 2$, as well as for $n \geqslant 3$, when restricted to the archimedean spherical spectrum.*

In addition, we address related counting and equidistribution problems and prove uniform Weyl laws (with explicit savings in the level aspect), estimates on the size of complementary spectrum, and uniform estimates on terms appearing in Arthur's trace formula for GL_n.