ISSN 0012-9593

quatrième série - tome 57 *fascicule 6* *novembre-décembre 2024*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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45, rue d'Ulm, 75230 Paris Cedex 05, France.
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Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 480 euros.
Abonnement avec supplément papier :
Europe : 675 €. Hors Europe : 759 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 57, 2024, p. 1599 à 1691

BLOW-UP DYNAMICS FOR SMOOTH FINITE ENERGY RADIAL DATA SOLUTIONS TO THE SELF-DUAL CHERN-SIMONS-SCHRÖDINGER EQUATION

BY KIHYUN KIM, SOONSIK KWON AND SUNG-JIN OH

ABSTRACT. – We consider the finite-time blow-up dynamics of solutions to the self-dual Chern-Simons-Schrödinger (CSS) equation (also referred to as the Jackiw-Pi model) near the radial soliton Q with the least L^2-norm (ground state). While a formal application of pseudoconformal symmetry to Q gives rise to an L^2-continuous curve of initial data sets whose solutions blow up in finite time, they all have infinite energy due to the slow spatial decay of Q. In this paper, we exhibit initial data sets that are smooth finite energy radial perturbations of Q, whose solutions blow up in finite time. It turns out that their blow-up rate differs from the pseudoconformal rate by a power of logarithm. Applying pseudoconformal symmetry in reverse, this also yields a first example of an infinite-time blow-up solution, whose blow-up profile contracts at a logarithmic rate.

Our analysis builds upon the ideas of previous works of the first two authors on (CSS) as well as celebrated works on energy-critical geometric equations by Merle, Raphaël, and Rodnianski. A notable feature of this paper is a systematic use of nonlinear covariant conjugations by the covariant Cauchy-Riemann operators in all parts of the argument. This not only overcomes the nonlocality of the problem, which is the principal challenge for (CSS), but also simplifies the structure of nonlinearity arising in the proof.

RÉSUMÉ. – Nous considérons la dynamique explosive en temps fini des solutions de l'équation de Chern-Simons-Schrödinger (CSS) autoduale (également appelée modèle de Jackiw-Pi) près du soliton radial Q avec la plus petite norme L^2. Alors qu'une application formelle de la symétrie pseudo-conforme à Q donne une courbe L^2-continue d'ensembles de données initiales dont les solutions explosent en temps fini, ils ont tous une énergie infinie en raison de la décroissance spatiale lente de Q. Dans cet article, nous obtenons des ensembles de données initiales qui sont des perturbations régulières radiales d'énergie finie de Q, dont les solutions explosent en temps fini. Il s'avère que leur taux de concentration diffère du taux pseudo-conforme par une puissance du logarithme. En appliquant la symétrie pseudo-conforme en sens inverse, cela donne également le premier exemple de solution explosive en temps infini, dont le profil d'explosion se contracte à un taux logarithmique.

Notre analyse s'appuie sur les idées développées antérieurement pour (CSS) par les deux premiers auteurs, ainsi que sur les célèbres travaux de Merle, Raphaël et Rodnianski sur les équations géométriques énergie-critique. Une caractéristique notable de cet article est l'utilisation systématique de conjugaisons non linéaires et covariantes par les opérateurs covariants de Cauchy-Riemann. Cela permet de surmonter non seulement la non-localité du problème, principal difficulté de (CSS), mais simplifie aussi la structure des non-linéarités apparaissant dans la preuve.

doi:10.24033/asens.2596

1. Introduction

The subject of this paper is the nonrelativistic Chern-Simons gauge field theory introduced by Jackiw-Pi [18], which is a Lagrangian field theory with the action

$$\mathcal{S}[\phi, A] := \frac{1}{2}\int_{\mathbb{R}^{1+2}} A \wedge F + \int_{\mathbb{R}^{1+2}} \frac{1}{2}\mathrm{Im}\,(\bar{\phi}\mathbf{D}_t\phi) + \frac{1}{2}|\mathbf{D}_x\phi|^2 - \frac{g}{4}|\phi|^4\, dt dx, \tag{1.1}$$

where $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is a complex-valued scalar field, $\mathbf{D}_\alpha = \partial_\alpha + iA_\alpha$ ($\alpha = t, 1, 2$) are the covariant derivatives associated with a real-valued 1-form $A = A_t dt + A_1 dx^1 + A_2 dx^2$ (connection 1-form) and $F = dA$ is the corresponding curvature 2-form. Note that (1.1) is simply the sum of the *Chern-Simons action*, $\frac{1}{2}\int A \wedge F$, and the action for the (gauge-covariant) cubic nonlinear Schrödinger equation. Following a widespread usage in the mathematical literature, we will refer the resulting Euler-Lagrange equation, written below in Section 1.1, as the *Chern-Simons-Schrödinger equation*.

The Chern-Simons action has been employed in high energy physics and condensed matter physics to describe interesting planar physics, such as topological massive gauge theories and the quantum Hall effect; we refer to [17, 18, 19, 20] for detailed reviews. The model (1.1) under consideration is of particular interest as it is the simplest model that is nonrelativistic (which is the setting of condensed matter physics) and, after a particular choice of the coupling constant g (namely $g = 1$), *self-dual*. A remarkable consequence of the self-duality, which was observed in the seminal paper of Jackiw-Pi [18], is the existence of explicit (!) spatially-localized static solutions to the model (also referred to as *solitons* or *nontopological vortices*) that are parametrized by the solutions to the (explicitly solvable) Liouville equation. In what follows, we refer to these solutions as *Jackiw-Pi vortices*.

Most basic among the Jackiw-Pi vortices is the *ground state* $(\mathbf{Q}, A)$, given in the polar coordinates (r, θ) by

$$\mathbf{Q}(r,\theta) = \sqrt{8}\frac{1}{1+r^2}, \quad A_t = \frac{1}{2}|\mathbf{Q}|^2, \quad A_r = 0, \quad A_\theta = -2\frac{r^2}{1+r^2}, \tag{1.2}$$

which has the minimal charge (i.e., the integral of $|\mathbf{Q}|^2$) among all Jackiw-Pi vortices. The charge is a natural measure of the size of a solution, as it is invariant under the scaling symmetry of (1.1). The ground state $\mathbf{Q}$ plays a pivotal role in the dynamics of solutions. Indeed, within radial symmetry, it is known that the L^2-norm of $\mathbf{Q}(x)$ serves as the threshold for global regularity and scattering [27]. An outstanding problem, then, is *to understand the dynamics of solutions associated to initial data in the vicinity of $\mathbf{Q}(x)$, with the L^2-norm greater than or equal to that of $\mathbf{Q}(x)$.*

In this regime, an interesting formal dynamics describing finite-time blow-up follows from the pseudoconformal symmetry of (1.3). Like the well-known cubic NLS on $\mathbb{R}^{1+2}$, the Chern-Simons-Schrödinger equation is invariant under the pseudoconformal transformations

$$(t,x) = \left(\tfrac{T}{1-bT}, \tfrac{X}{1-bT}\right), \qquad \Phi_b(T,X) = \tfrac{1}{1-bT}e^{-ib\frac{|X|^2}{1-bT}}\phi\left(\tfrac{T}{1-bT}, \tfrac{X}{1-bT}\right),$$

where $b \in \mathbb{R}$. Applying such transformations with $b > 0$ to the ground state, we obtain a one parameter family of solutions $(\mathbf{S}_b, A_b)$ blowing up in finite time (namely, at $T = b^{-1}$). Each $\mathbf{S}_b$ has the same L^2-norm as $\mathbf{Q}$ and $\mathbf{S}_b(t = 0) \to \mathbf{Q}$ in L^2 as $b \to 0+$. However, because of the slow spatial decay of $\mathbf{Q}$, each $\mathbf{S}_b$ ($b > 0$) has *infinite* $\dot{H}^1$-norm (as well as infinite conserved energy, which is defined below). As a result, if we consider the dynamics of *finite energy* solutions in the vicinity of $\mathbf{Q}$, the relevance of $\mathbf{S}_b$ and even the possibility of a finite-time blow-up are dubious (1).

The main result of this paper is the first construction of finite time blow-up solutions with smooth finite energy radial initial data, which are arbitrarily close to $\mathbf{Q}$ in the L^2-topology. A detailed description of the blow-up dynamics is given; in particular, we provide a codimension one set of data leading to the blow-up, as well as a sharp description of the rate. The blow-up rate differs from the pseudoconformal rate by a factor of logarithm. This is a sharp contrast to the case of higher equivariance indices $m \geq 1$, in which case the pseudoconformal blow-up rate is obtained [22]. Interestingly, our blow-up rate is identical to the one obtained in the 1-equivariant Schrödinger maps [33]. Via the pseudoconformal transform, we also construct infinite-time blow-up solutions with the blow-up profile $\mathbf{Q}$, whose scale contracts at a rate logarithmic in t.

Our analysis follows the road map furnished by the seminal works of Rodnianski-Sterbenz [41], Raphaël-Rodnianski [38], and Merle-Raphaël-Rodnianski [33] in the cases of wave maps, Yang-Mills, and Schrödinger maps. Compared to the previously considered cases, a key challenge in the Chern-Simons-Schrödinger case is the nonlocality of the nonlinearity, which results in a stronger soliton-radiation interaction. Notable features of our proof are a systematic use of nonlinear covariant conjugations, and the treatment of the self-dual Chern-Simons-Schrödinger equation as a coupled system of nonlinearly conjugated variables of varying orders. These ideas provide a simple and efficient way to overcome the nonlocality of the problem. This point of view pervades all steps of our arguments, such as the derivation of modified profiles and sharp modulation laws, decomposition of solutions, and energy estimates. See Section 1.4 for more details.

1.1. The self-dual Chern-Simons-Schrödinger equation

The Euler-Lagrange equation for (1.1) in the self-dual case $g = 1$ takes the form

$$\begin{cases} \mathbf{D}_t\phi = i(\mathbf{D}_1\mathbf{D}_1 + \mathbf{D}_2\mathbf{D}_2)\phi + i|\phi|^2\phi, \\ F_{t1} = -\mathrm{Im}\,(\overline{\phi}\mathbf{D}_2\phi), \\ F_{t2} = \mathrm{Im}\,(\overline{\phi}\mathbf{D}_1\phi), \\ F_{12} = -\tfrac{1}{2}|\phi|^2. \end{cases} \tag{1.3}$$

We remind the reader that $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is a complex-valued scalar field, $\mathbf{D}_\alpha = \partial_\alpha + iA_\alpha$ ($\alpha = t, 1, 2$) are the covariant derivatives associated with a real-valued 1-form $A = A_t dt + A_1 dx^1 + A_2 dx^2$ (connection 1-form) and $F = dA$ is the corresponding curvature 2-form. We will refer to this equation as the (self-dual) *Chern-Simons-Schrödinger* (CSS) equation.

(1) Another standard method to deduce finite-time blow-up is using the virial identity à la Glassey, but in the self-dual case, it only leads to a pseudoconformal transform of a static solution; see [23].