ISSN 0012-9593

quatrième série - tome 57 *fascicule 6* *novembre-décembre 2024*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Christian BÄR & Alexander STROHMAIER

Local Index Theory for Lorentzian Manifolds

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 480 euros.
Abonnement avec supplément papier :
Europe : 675 €. Hors Europe : 759 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 57, 2024, p. 1693 à 1752

LOCAL INDEX THEORY FOR LORENTZIAN MANIFOLDS

BY CHRISTIAN BÄR AND ALEXANDER STROHMAIER

ABSTRACT. – Index theory for Lorentzian Dirac operators is a young subject with significant differences to elliptic index theory. It is based on microlocal analysis instead of standard elliptic theory and one of the main features is that a nontrivial index is caused by topologically nontrivial dynamics rather than nontrivial topology of the base manifold. In this paper we establish a local index formula for Lorentzian Dirac-type operators on globally hyperbolic spacetimes. This local formula implies an index theorem for general Dirac-type operators on spatially compact spacetimes with Atiyah-Patodi-Singer boundary conditions on Cauchy hypersurfaces. This is significantly more general than the previously known theorems that require the compatibility of the connection with Clifford multiplication and the spatial Dirac operator on the Cauchy hypersurface to be selfadjoint with respect to a positive definite inner product.

RÉSUMÉ. – La théorie de l'indice pour les opérateurs de Dirac lorentziens est un sujet récent qui présente des différences importantes avec la théorie d'indice elliptique. Elle est basée sur l'analyse microlocale au lieu de la théorie elliptique standard et l'une de ses principales caractéristiques est qu'un indice non trivial est causé par une dynamique topologiquement non triviale plutôt que par la topologie non triviale de la variété de base. Dans cet article, nous obtenons une formule locale d'indice pour les opérateurs lorentziens de type Dirac sur les espaces-temps globalement hyperboliques. Cette formule locale implique un théorème d'indice pour les opérateurs généraux de type Dirac sur des espaces-temps spatialement compacts avec des conditions limites d'Atiyah-Patodi-Singer sur les hypersurfaces de Cauchy. Il s'agit d'un théorème beaucoup plus général que les théorèmes connus précédemment, qui exigent la compatibilité de la connexion avec la multiplication de Clifford et que l'opérateur de Dirac spatial sur l'hypersurface de Cauchy soit auto-adjoint par rapport à un produit scalaire défini positif.

Introduction

The Atiyah-Singer index theorem [2] is one of the most important achievements of mathematics in the 20th century. It provides many conceptual insights by identifying topological invariants as indices of geometrically defined differential operators, thereby linking geometry, topology, and analysis. The index theorem is most conveniently stated for a twisted

 doi:10.24033/asens.2597

Dirac operator $\not{D}$ on a closed Riemannian manifold, and it then gives a formula for the index as an integral over a density that is determined by local geometric quantities. In fact, there is a local version of the index theorem that expresses the difference of the local traces of two heat kernels associated to the Laplace-type operators $\not{D}\not{D}^*$ and $\not{D}^*\not{D}$ in the limit as time goes to zero. This local index formula implies the Atiyah-Singer index theorem on a closed Riemannian manifold by the McKean-Singer formula. We refer to the monograph [10] for details and further references.

It is the local index theorem that allows the more general treatment of operators in the L^2-settings, in relative settings, and on manifolds with boundary. As an example we mention Atiyah's theorem on the equality of the index and the L^2-index on the universal cover [1], the proof of which relies heavily on the local index theorem. For manifolds with boundary an index formula was given by Atiyah, Patodi, and Singer, see MR397797, MR397798, MR397799 or also [36]. This formula contains a boundary contribution which gave rise to the study of the η-invariant and all its applications. Local index theory also established the link between Quillen's theory of superconnections with the index theorem for families [12]. There are further generalizations of the index theorem dealing with elliptic operators on manifolds with singularities, noncompact manifolds, or hypoelliptic operators, see for example MR730920,MR720933,MR1876286,MR1156670,MR2680395 to mention only a few.

Developing index theory for Lorentzian manifolds seems hopeless at first, since Dirac-type operators are hyperbolic in this case. On a closed manifold an operator needs to be elliptic to be Fredholm. So there is no Lorentzian analog to the Atiyah-Singer index theorem. Surprisingly, the situation changes in the presence of boundary. In [9] we proved a Lorentzian index theorem for Dirac operators on globally hyperbolic spacetimes with appropriate boundary conditions imposed in the timelike future and past. This is an analog to the Atiyah-Patodi-Singer index theorem in the Riemannian setting. In fact, the boundary conditions and the geometric formula for the index are exactly the same as in the Riemannian setting.

This allows for the direct application of index theory in situations relevant in relativistic physics, when separation of variables and reduction to an elliptic system or Wick rotation is not possible. For example, the chiral anomaly in quantum field theory can be understood as such an index [8].

The Lorentzian index theorem was extended to more general boundary conditions by the first author and Hannes in [7], to the noncompact Callias setting by Braverman in [15], and to a noncompact Γ-equivariant setting by Damaschke in [17]. Parts of the theorem have also been generalized to a more abstract framework by van den Dungen and Ronge in ronge,dungen. In [44], Shen and Wrochna replace the timelike compact dynamics by asymptotic conditions for infinite times.

The fact that this hyperbolic operator is Fredholm is in all these cases not based on elliptic theory, but rather relies on Hörmander's propagation of singularities theorem. This is reminiscent of the use of propagation estimates by Hörmander showing finite dimensionality of the space of solutions of operators of real principal type, see [21]. We mention that refined propagation of singularities is also a key behind the Fredholm theory introduced by Vasy [47], which has found many applications.

The present paper extends the Lorentzian index theorem in two directions:

(1) We drop the assumption of a positive definite inner product on the vector bundle along the spacelike hypersurfaces with respect to which the induced spatial Dirac operator is formally selfadjoint. This requirement is indeed not natural from a Lorentzian point of view and it is not satisfied by many geometric operators. Dropping this assumption, one can no longer reduce to the Riemannian APS-index theorem using spectral flow arguments as in [9]. The deformation argument that was used there is manifestly nonlocal and relies heavily on the invariance properties of characteristic classes. A local proof cannot be carried out in that way. Instead, we will give a purely Lorentzian proof and the Riemannian index theorem by Atiyah, Patodi, and Singer is no longer used. This also allows us to treat the significantly larger class of general Dirac-type operators instead of certain twisted spinorial Dirac operators only.

(2) We provide a local interpretation of the index in the Lorentzian context, and we show that a local index formula holds for this quantity, also in the context of spatially noncompact spacetimes. The role played by the heat kernel in the Riemannian setting is now taken over by Feynman parametrices. The short-time asymptotics of the heat kernel is replaced by the Hadamard expansion. We show that the index of the Lorentzian Dirac operator is the integral of a locally defined current over the Cauchy hypersurface. This can be seen as the Lorentzian analog of the McKean-Singer formula. The η-invariant appears naturally in this context. We note that the treatment of analogous currents using the Hadamard expansion has appeared in the physics literature on quantum field theory on curved spacetimes (see the work by Zahn [49]) and our approach also generalizes these to Dirac-type operators.

Due to the fundamental role played by Feynman parametrices we study them in quite some detail. We give a detailed construction that is similar to that of the classical Hadamard parametrix but is based on a family of distributions with distinguished microlocal properties. This construction results in a similar expansion as the one for Hadamard states in the physics literature (see [32]) but is simpler and gives a more conceptual approach to the corresponding expansion for Feynman parametrices. Indeed, it has been known for a while that in even dimensions one has to add logarithmic terms to such an expansion (see for example [34] for a direct construction of the Feynman propagator or [50] for a nice survey in the ultrastatic case). In our treatment, the logarithmic terms appear naturally as derivatives, and the recursion formulae are obtained by simply differentiating the Hadamard recursion formulae. More concretely, a special case of Proposition A.9 says

PROPOSITION. – *Let P be a normally hyperbolic operator acting on sections of a vector bundle $\mathscr{S}$ over a $2m$-dimensional globally hyperbolic manifold X.*

Then every Feynman parametrix G of P is of the form

$$G = G^{\mathrm{loc}} + G^{\mathrm{reg}}$$