ISSN 0012-9593

quatrième série - tome 57 fascicule 6 novembre-décembre 2024

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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45, rue d'Ulm, 75230 Paris Cedex 05, France.
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Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Email : `abonnements@smf.emath.fr`

Tarifs

Abonnement électronique : 480 euros.
Abonnement avec supplément papier :
Europe : 675 €. Hors Europe : 759 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 57, 2024, p. 1753 à 1843

CONVEX COCOMPACT ACTIONS IN REAL PROJECTIVE GEOMETRY

BY Jeffrey DANCIGER, François GUÉRITAUD AND Fanny KASSEL

Abstract. – We study a notion of convex cocompactness for discrete subgroups of the projective general linear group acting (not necessarily irreducibly) on real projective space, and give various characterizations. A convex cocompact group in this sense need not be word hyperbolic, but we show that it still has some of the good properties of classical convex cocompact subgroups in rank-one Lie groups. Extending our earlier work from the context of projective orthogonal groups, we show that for word hyperbolic groups preserving a properly convex open set in projective space, the above general notion of convex cocompactness is equivalent to a stronger convex cocompactness condition studied by Crampon-Marquis, and also to the condition that the natural inclusion be a projective Anosov representation. We investigate examples.

Résumé. – Nous étudions et donnons diverses caractérisations d'une condition de convexe cocompacité pour les sous-groupes discrets du groupe projectif linéaire qui agissent (irréductiblement ou non) sur l'espace projectif réel. Bien qu'il ne soit pas toujours Gromov-hyperbolique, un groupe convexe cocompact en ce sens conserve certaines des bonnes propriétés des sous-groupes convexes cocompacts classiques dans les groupes de Lie de rang un. Nous étendons nos travaux antérieurs du contexte des groupes orthogonaux non compacts au groupe projectif linéaire en montrant que pour un groupe Gromov-hyperbolique préservant un ouvert proprement convexe de l'espace projectif, cette condition de convexe cocompacité est équivalente à une notion de convexe cocompacité plus forte développée par Crampon et Marquis, ainsi qu'à la condition que l'inclusion naturelle est une représentation anosovienne projective. Nous détaillons quelques exemples.

This project received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC starting grant DiGGeS, grant agreement No 715982). The authors also acknowledge support from the GEAR Network, funded by the National Science Foundation under grants DMS 1107452, 1107263, and 1107367 ("RNMS: GEometric structures And Representation varieties"). J.D. was partially supported by an Alfred P. Sloan Foundation fellowship, and by the National Science Foundation under grant DMS 1510254. F.G. and F.K. were partially supported by the Agence Nationale de la Recherche through the Labex CEMPI (ANR-11-LABX-0007-01) and the grant DynGeo (ANR-16-CE40-0025-01). Part of this work was completed while F.K. was in residence at the MSRI in Berkeley, California, for the program *Geometric Group Theory* (Fall 2016) supported by NSF grant DMS 1440140, and at the INI in Cambridge, UK, for the program *Nonpositive curvature group actions and cohomology* (Spring 2017) supported by EPSRC grant EP/K032208/1.

 doi:10.24033/asens.2598

1. Introduction

In the classical setting of semisimple Lie groups G of real rank one, a discrete subgroup of G is said to be convex cocompact if it acts cocompactly on some nonempty closed convex subset of the Riemannian symmetric space G/K of G. Such subgroups have been abundantly studied, in particular in the context of Kleinian groups and real hyperbolic geometry, where there is a rich world of examples. They are known to display good geometric and dynamical behavior.

Nevertheless, in higher-rank semisimple Lie groups G, the condition that a discrete subgroup Γ acts cocompactly on some nonempty convex subset of the Riemannian symmetric space G/K turns out to be quite restrictive: Kleiner-Leeb [52] and Quint [59] proved, for example, that if G is simple and such a subgroup Γ is Zariski-dense in G, then it is in fact a uniform lattice of G.

The notion of an *Anosov representation* of a word hyperbolic group in a higher-rank semisimple Lie group G, introduced by Labourie [54] and generalized by Guichard-Wienhard [43], is a much more flexible notion, which has earned a central role in higher Teichmüller-Thurston theory, see e.g., [14, 16, 18, 40, 49]. Anosov representations are defined, not in terms of convex subsets of the Riemannian symmetric space G/K, but rather in terms of a dynamical condition for the action on a certain flag variety, i.e., on a compact homogeneous space G/P. This dynamical condition guarantees many desirable analogies with convex cocompact subgroups in rank one: see e.g., [43, 49, 50, 51, 54]. It also allows for the definition of certain interesting geometric structures associated to Anosov representations: see e.g., [24, 40, 41, 43, 50]. However, natural *convex* geometric structures associated to Anosov representations have been lacking in general. Such structures could allow geometric intuition to bear more fully on Anosov representations, making them more accessible through familiar geometric constructions such as convex fundamental domains, and potentially unlocking new sources of examples. While there is a rich supply of examples of Anosov representations into higher-rank Lie groups in the case of surface groups or free groups, it has proven difficult to construct examples for more complicated word hyperbolic groups.

One of the goals of this paper is to show that, when $G = \mathrm{PGL}(\mathbb{R}^n)$ is a projective linear group, there are, in many cases, natural convex cocompact geometric structures modeled on $\mathbb{P}(\mathbb{R}^n)$ associated to Anosov representations into G. The idea is the following: to any discrete subgroup Γ of $G = \mathrm{PGL}(\mathbb{R}^n)$ are associated two *limit sets* Λ_Γ and Λ_Γ^*. Recall that an element $g \in \mathrm{PGL}(\mathbb{R}^n)$ is said to be *proximal* in the real projective space $\mathbb{P}(\mathbb{R}^n)$ if it admits a unique attracting fixed point in $\mathbb{P}(\mathbb{R}^n)$ (see Section 2.3). The *proximal limit set* Λ_Γ of Γ in $\mathbb{P}(\mathbb{R}^n)$ is defined as the closure of the set of attracting fixed points of proximal elements of Γ in $\mathbb{P}(\mathbb{R}^n)$. Similarly, we consider the proximal limit set Λ_Γ^* of Γ in the dual projective space $\mathbb{P}((\mathbb{R}^n)^*)$ for the dual action; we can view it as a set of projective hyperplanes in $\mathbb{P}(\mathbb{R}^n)$. Suppose the complement $\mathbb{P}(\mathbb{R}^n) \smallsetminus \bigcup_{H \in \Lambda_\Gamma^*} H$ is nonempty. Its connected components are open sets which (as soon as $\Lambda_\Gamma^* \neq \emptyset$) are *convex*, in the sense that they are contained and convex in some affine chart of $\mathbb{P}(\mathbb{R}^n)$; when Γ acts irreducibly on $\mathbb{P}(\mathbb{R}^n)$, these components are even *properly convex*, in the sense that they are convex and bounded in some affine chart. If one of these convex open sets, call it $\Omega_{\max}$, is properly convex and *invariant* under Γ, then the action of Γ on $\Omega_{\max}$ is necessarily properly discontinuous (see Section 2.1), the set Λ_Γ is contained in the boundary $\partial\Omega_{\max}$, and it makes sense to consider the convex hull of Λ_Γ in $\Omega_{\max}$: this is a closed convex subset $\mathcal{C}$ of $\Omega_{\max}$. When $\Gamma \hookrightarrow G$ is (projective) Anosov, we prove that the action of Γ on the convex set $\mathcal{C}$ is cocompact. Further, we prove that Γ satisfies a *stronger* notion of projective convex cocompactness introduced by Crampon-Marquis [27]. Conversely, we show that convex cocompact subgroups of $\mathbb{P}(\mathbb{R}^n)$ in the sense of [27] always give rise to Anosov representations, which enables us to give new examples of Anosov representations and study their deformation spaces by constructing these geometric structures directly. In [31] we had previously established this close connection between convex cocompactness in projective space and Anosov representations in the case of irreducible representations valued in a projective orthogonal group $\mathrm{PO}(p,q)$.

One context where such a connection between Anosov representations and convex projective structures has been known for some time is the deformation theory of real projective surfaces, for $G = \mathrm{PGL}(\mathbb{R}^3)$ [21, 39]. More generally, it follows from work of Benoist [8] that if a discrete subgroup Γ of $G = \mathrm{PGL}(\mathbb{R}^n)$ *divides* (i.e., acts cocompactly on) a strictly convex open subset Ω of $\mathbb{P}(\mathbb{R}^n)$, then Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow G$ is Anosov. In this particular case Λ_Γ is the boundary of Ω and Λ_Γ^* is the collection of supporting hyperplanes to Ω.

Benoist [10] also found examples of discrete subgroups of $\mathrm{PGL}(\mathbb{R}^n)$, acting irreducibly on $\mathbb{P}(\mathbb{R}^n)$, which divide properly convex open sets that are not strictly convex, for $4 \leq n \leq 7$; these subgroups are not word hyperbolic. In this paper we study a broad notion of convex cocompactness for discrete subgroups Γ of $\mathrm{PGL}(\mathbb{R}^n)$ acting on $\mathbb{P}(\mathbb{R}^n)$ which simultaneously generalizes Crampon-Marquis's notion and Benoist's divisible convex sets [7, 8, 9, 10]. While we mainly take the point of view of examining limit sets and convex hulls in projective space, we also show that this notion of convex cocompactness is characterized by the property that Γ is (up to finite index) the holonomy group of a compact convex projective manifold with strictly convex boundary. Cooper-Long-Tillmann [26] have studied the deformation theory of such manifolds and their work implies that this notion is stable under small deformations of Γ in $\mathrm{PGL}(\mathbb{R}^n)$. We show further that it is stable under deformation into larger projective general linear groups $\mathrm{PGL}(\mathbb{R}^{n+n'})$, after the model of quasi-Fuchsian deformations of Fuchsian groups. This yields examples of nonhyperbolic discrete subgroups which satisfy our convex cocompactness property but do not divide a properly convex open set.

We now describe our results in more detail.

1.1. Strong convex cocompactness in $\mathbb{P}(V)$ and Anosov representations

In the whole paper, we fix an integer $n \geq 2$ and set $V := \mathbb{R}^n$. Recall that an open subset Ω of the projective space $\mathbb{P}(V)$ is called *convex* if it is contained and convex in some affine chart, *properly convex* if its closure is convex, and *strictly convex* if in addition its boundary does not contain any nontrivial projective line segment. It is said to have *boundary of class C^1* (or just *C^1 boundary*) if every point of the boundary of Ω has a unique supporting hyperplane.

In [27], Crampon-Marquis introduced a notion of *geometrically finite action* of a discrete subgroup Γ of $\mathrm{PGL}(V)$ on a strictly convex open domain of $\mathbb{P}(V)$ with boundary of class C^1. If cusps are not allowed (or equivalently, if we request all infinite-order elements to be