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E_∞-CELLS AND GENERAL LINEAR GROUPS OF FINITE FIELDS

BY Søren GALATIUS, Alexander KUPERS
AND Oscar RANDAL-WILLIAMS

ABSTRACT. – We prove new homological stability results for general linear groups over finite fields. These results are obtained by constructing CW approximations to the classifying spaces of these groups, in the category of E_∞-algebras, guided by computations of homology with coefficients in the E_1-split Steinberg module.

RÉSUMÉ. – Nous prouvons de nouveaux résultats de stabilité homologique pour les groupes généraux linéaires sur les corps finis. Ces résultats sont obtenus en construisant des approximations cellulaires du classifiant de ces groupes dans la catégorie des algèbres E_∞, guidées par des calculs d'homologie à coefficients dans le module de Steinberg E_1-scindé.

1. Introduction

The homology of general linear groups over finite fields was studied by Quillen in his seminal paper on the K-theory of finite fields [19, Theorem 3]. One of his main results is the complete determination of the homology groups $H_d(\mathrm{GL}_n(\mathbb{F}_q);\mathbb{F}_\ell)$ where $\mathbb{F}_q$ is a finite field with $q = p^r$ elements, $\mathrm{GL}_n(\mathbb{F}_q)$ denotes the general linear group of the n-dimensional vector space $\mathbb{F}_q^n$, and $\ell \neq p$.

When $\ell = p$ full information about these homology groups is only available in a stable range. Extending linear isomorphisms of $\mathbb{F}_q^{n-1}$ to $\mathbb{F}_q^n = \mathbb{F}_q^{n-1} \oplus \mathbb{F}_q$ by the identity on the second subspace induces an injective homomorphism

$$s\colon \mathrm{GL}_{n-1}(\mathbb{F}_q) \longrightarrow \mathrm{GL}_n(\mathbb{F}_q)$$

called the *stabilization map*. Forming the colimit over such stabilization maps, Quillen has proved [19, Corollary 2] that

$$H_d(\mathrm{GL}_\infty(\mathbb{F}_q);\mathbb{F}_p) = 0 \text{ for } d > 0, \tag{1.1}$$

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which has implications for finite n using homological stability. The best homological stability result available in the literature before the writing of this paper is due to Maazen, who proved that

$$H_d(\mathrm{GL}_n(\mathbb{F}_q), \mathrm{GL}_{n-1}(\mathbb{F}_q); \mathbb{F}_p) = 0$$

for $d < \frac{n}{2}$ [17]. It is an unpublished result of Quillen that for $q \neq 2$ these groups vanish in the larger range $d < n$. (A proof appears in his unpublished notebooks [21, 1974-I, p. 10]. The first pages of this notebook were unfortunately bleached by sunlight, which makes it difficult to follow the argument. A recent paper of Sprehn and Wahl [26] include an exposition of Quillen's argument, as well as a generalization to other classical groups.)

1.1. Homological stability

In this paper we study the homology of general linear groups of finite fields of characteristic p, with coefficients in $\mathbb{F}_p$, using E_k-cellular approximations. We developed this technique in [9], and in [8] we applied it to studying homology of mapping class groups. The case of infinite fields is treated in [10]. For fields with more than two elements we obtain the following strengthening of Quillen's result (see also [26, Theorem A]).

Theorem A. – *If $q = p^r \neq 2$, then*

$$H_d(\mathrm{GL}_n(\mathbb{F}_q), \mathrm{GL}_{n-1}(\mathbb{F}_q); \mathbb{F}_p) = 0 \text{ for } d < n + r(p-1) - 2.$$

The bound $n + r(p-1) - 2$ is linear of slope 1 as was Quillen's, but our offset $r(p-1)-2$ is strictly positive for $q > 4$. Our methods also apply to the field $\mathbb{F}_2$, where we obtain the following linear bound of slope $2/3$.

Theorem B. – *We have $H_d(\mathrm{GL}_n(\mathbb{F}_2), \mathrm{GL}_{n-1}(\mathbb{F}_2); \mathbb{F}_2) = 0$ for $d < \frac{2(n-1)}{3}$.*

Combining this with Quillen's calculation (1.1) (or using Corollary 6.2 when $q = 2$), we obtain the following vanishing theorem.

Corollary C. – *If $q = p^r \neq 2$, then $\widetilde{H}_d(\mathrm{GL}_n(\mathbb{F}_q); \mathbb{F}_p) = 0$ for $d+1 < n + r(p-1) - 1$. If $q = 2$, then $\widetilde{H}_d(\mathrm{GL}_n(\mathbb{F}_2); \mathbb{F}_2) = 0$ for $d + 1 < \frac{2n}{3}$.*

1.2. A question of Milgram and Priddy

In [18], Milgram and Priddy constructed a cohomology class $\det_n \in H^n(M_{n,n}(\mathbb{F}_2); \mathbb{F}_2)$, where $M_{n,n}(\mathbb{F}_2) \subset \mathrm{GL}_{2n}(\mathbb{F}_2)$ is the subgroup of $(2n \times 2n)$-matrices of the form

$$\begin{bmatrix} \mathrm{id}_n & * \\ 0 & \mathrm{id}_n \end{bmatrix},$$

which is invariant for the natural action of $\mathrm{GL}_n(\mathbb{F}_2) \times \mathrm{GL}_n(\mathbb{F}_2)$ as the subgroup of block-diagonal matrices. They suggested (see also [3, Section 5]) that this class may be in the image under the restriction map

$$H^*(\mathrm{GL}_{2n}(\mathbb{F}_2); \mathbb{F}_2) \longrightarrow H^*(M_{n,n}(\mathbb{F}_2); \mathbb{F}_2)^{\mathrm{GL}_n(\mathbb{F}_2) \times \mathrm{GL}_n(\mathbb{F}_2)}.$$

By Maazen's result it would then be the lowest-degree such class. They showed that this is indeed the case for $\det_1$ and $\det_2$. However Corollary C implies that $H_n(\mathrm{GL}_{2n}(\mathbb{F}_2); \mathbb{F}_2) = 0$ for $n > 3$, so that $\det_n$ *cannot* be in the image of the restriction map for $n > 3$. In Section 6.3

we combine our techniques with a recent computation of Szymik to prove that $\det_3$ does lie in the image of the restriction map. This completely answers the question of Milgram and Priddy.

1.3. Homology of Steinberg modules

The reduced top integral homology of the Tits building associated to $\mathbb{F}_q^n$ is equal to the Steinberg module $\mathrm{St}(\mathbb{F}_q^n)$. It has an action of $\mathrm{GL}_n(\mathbb{F}_q)$, and in the modular representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$ a central role is played by the $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_q)]$-module $\mathrm{St}(\mathbb{F}_q^n) \otimes \mathbb{F}_p$, as it is the only module which is both irreducible and projective. This implies that $H_*(\mathrm{GL}_n(\mathbb{F}_q); \mathrm{St}(\mathbb{F}_q^n) \otimes \mathbb{F}_p) = 0$. Combining the methods of this paper with Quillen's calculation of the cohomology of $\mathrm{GL}_n(\mathbb{F}_q)$ away from the defining characteristic, we are also able to calculate the groups $H_*(\mathrm{GL}_n(\mathbb{F}_q); \mathrm{St}(\mathbb{F}_q^n) \otimes \mathbb{F}_\ell)$ for $\ell \neq p$. We give this calculation in Section 7.

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2. Recollections on a homology theory for E_∞-algebras

In this section we informally explain that part of the theory developed in [9] is necessary to prove Theorems A and B. We refer to that paper for a more formal discussion as well as for proofs; we shall refer to things labeled X in [9] as $E_k.X$ throughout this paper. The reader may find it helpful to consult Sections 3 and 5 of [8], which contain applications of this theory to mapping class groups that are similar in technique to the applications in this paper.

The aforementioned theory concerns CW approximation for E_∞-algebras. Because we are interested mainly in the *homology* of certain E_∞-algebras, we will work in the category of *simplicial k-modules*. In this paper k shall always be a field, and we will also make this simplifying assumption in this section. In the case where $k = \mathbb{Q}$, an E_∞-algebra in simplicial k-modules is equivalent to the data of a commutative algebra in non-negatively graded chain complexes over $\mathbb{Q}$.

In order to eventually keep track of the dimension n of the vector space $\mathbb{F}^n$ we shall work with an additional $\mathbb{N}$-grading (for us $\mathbb{N}$ always includes 0) which we call *rank*, and so work in the category $\mathsf{sMod}_k^{\mathbb{N}}$ of $\mathbb{N}$-graded simplicial k-modules. Morphisms in this category are weak equivalences if they are weak equivalences of underlying $\mathbb{N}$-graded simplicial sets, and the category is equipped with the symmetric monoidal structure given by Day convolution.