ISSN 0012-9593

quatrième série - tome 57 fascicule 6 novembre-décembre 2024

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Anna BOT

A smooth complex rational affine surface with uncountably many real forms

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 480 euros.
Abonnement avec supplément papier :
Europe : 675 €. Hors Europe : 759 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 57, 2024, p. 1883 à 1897

A SMOOTH COMPLEX RATIONAL AFFINE SURFACE WITH UNCOUNTABLY MANY REAL FORMS

BY ANNA BOT

ABSTRACT. – We exhibit a smooth complex rational affine surface with uncountably many nonisomorphic real forms.

RÉSUMÉ. – Nous présentons une surface affine rationnelle complexe lisse avec un ensemble indénombrable de formes réelles non isomorphes.

1. Introduction

The study of real forms of a complex algebraic variety has seen substantial progress: the finiteness of isomorphism classes over $\mathbb{R}$ of real forms has been proven for example for abelian varieties [6, 28], projective algebraic surfaces of Kodaira dimension greater than or equal to one [10, Appendix D], minimal projective algebraic surfaces [10, Appendix D], del Pezzo surfaces [27] and compact hyperkähler manifolds [9]. Lesieutre constructed in [25] the first example of a complex projective variety with infinitely many nonisomorphic real forms, and the construction inspired the examples of complex projective varieties in every dimension $d \geq 2$ of Kodaira dimension $d-2$ by Dinh and Oguiso [11], and the ones of smooth rational projective varieties of dimension greater than or equal to three by Dinh, Oguiso and Yu [12].

Using a field-theoretic approach, [24] proves that any projective variety can have at most countably many real forms. We give a short proof in the particular case of rational projective surfaces in Proposition 2.5. Yet, in the affine case, a priori no restrictions hold on the number of real forms. However, such varieties may only arise in dimension at least 2—see Proposition 2.4 for a proof of why smooth affine curves admit at most finitely many real forms. Dubouloz, Freudenburg and Moser-Jauslin [14] constructed rational affine varieties of dimension greater or equal to four with at least countably infinitely many pairwise

The author acknowledges support by the Swiss National Science Foundation Grant "Geo-metrically ruled surfaces" 200020-192217.

nonisomorphic real forms. This prompts the question: can there exist a rational affine surface with uncountably many pairwise nonisomorphic real forms? We answer in the affirmative, and even describe a moduli space which parametrises an uncountable collection of nonisomorphic real forms of the suitable affine surface:

THEOREM 1.1. – *There exist a real affine variety $\mathcal{X}$ and a flat morphism $p: \mathcal{X} \to \mathbb{A}^1_{\mathbb{R}} \setminus \{0,1\}$ defined over $\mathbb{R}$ satisfying the following properties:*

1. *The closed fibers of $p_{\mathbb{C}}: \mathcal{X}_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}} \setminus \{0,1\}$ are pairwise isomorphic smooth complex rational affine surfaces.*
2. *The fibers of p over $\mathbb{R}$-rational closed points are pairwise birationally diffeomorphic smooth real rational affine surfaces.*
3. *Two fibers $p^{-1}(\alpha)$, $p^{-1}(\beta)$ with $\alpha, \beta \in \mathbb{R} \setminus \{0,1\}$ are isomorphic if and only if $\alpha\beta = 1$ or $\alpha = \beta$.*

Statement 2 implies that the nonisomorphic real forms appearing in this moduli space arise for algebraic reasons only and not due to some topological obstruction. Also, choosing only $\mathbb{Q}$-rational closed points $\alpha \in \mathbb{A}^1_{\mathbb{R}}(\mathbb{Q})$, we find infinitely (yet countably!) many $\mathbb{Q}$-forms using the above construction; the same holds for any subfield of $\mathbb{R}$. We further point out that the theory and the proofs would also work for a real closed field $\mathbf{k}$ and its algebraic closure $\overline{\mathbf{k}}$ in lieu of $\mathbb{R}$ and $\mathbb{C}$, to the effect that the classes of nonisomorphic $\mathbf{k}$-forms constitute a set of cardinality $|\mathbf{k}|$.

The question whether there exists a smooth complex rational projective surface with an infinite number of isomorphism classes of real forms was posed for example by [10, pages 232-233], [1, Problem, page 1128], [11, page 943] or [12, Question 1.5], and an unboundedness result was given in [7]. Dinh, Oguiso and Yu [13] provided an example of a smooth complex rational projective surface with infinitely many pairwise nonisomorphic real forms, using a nifty construction involving the Kummer K3 surface associated to the product of two elliptic curves. With this, rational surfaces with as many pairwise nonisomorphic real forms as possible have been found, both in the affine and the projective case.

The complex rational affine surface constituting a closed fiber $p_{\mathbb{C}}^{-1}(\alpha)$ of the fibration in Theorem 1.1 1 with $\alpha \in \mathbb{C} \setminus \{0,1\}$ can be given explicitly as the blow-up of $\mathbb{A}^2_{\mathbb{C}}$ in the points $(0,0)$, $(1,0)$, $(\alpha,0)$, $(0,1)$ and $(0,\alpha)$, with the strict transforms of the lines $x = 0$ and $y = 0$ removed, see Section 3.2. Such an affine surface can be completed by a linear chain of smooth rational curves and is therefore called a *Gizatullin surface* (see Section 3)—in this case, the curves are the strict transforms of $x = 0$ and $y = 0$ as well as the line at infinity. In fact, the results in this text were first obtained using the description of the automorphism group of the Gizatullin surface given in [5] by Blanc and Dubouloz.

Most likely, the real forms we exhibit in this article are by far not all one can construct, so it would be interesting to determine:

QUESTION 1.2. – *Are there real forms of a closed fiber of $p_{\mathbb{C}}$ other than $p^{-1}(\alpha)$ for $\alpha \in \mathbb{R} \setminus \{0,1\}$, and can one find some not birationally diffeomorphic to $p^{-1}(\alpha)$?*

As real forms are a special case of twisted forms, the following questions pose themselves.

QUESTION 1.3. – *Fix a field* $\mathbf{k}$ *with separable closure* $\mathbf{k}_s$. *Is it possible to construct infinitely/uncountably many affine varieties* X *defined over* $\mathbf{k}$ *and pairwise nonisomorphic over* $\mathbf{k}$ *such that their base changes* $X \times_{\mathrm{Spec}(\mathbf{k})} \mathrm{Spec}(\mathbf{k}_s)$ *are all isomorphic over* $\mathbf{k}_s$*?*

QUESTION 1.4. – *What about the special case of* $\mathbf{k} = \mathbb{Q}_p$*? What about* $\mathbf{k} = \mathbb{C}(t)$*?*

The paper is structured as follows: we start by introducing some preliminary notions as well as the correspondence between complex schemes with a real structure and real schemes in Section 2. Then, we describe the moduli space in Section 3, with two isomorphic descriptions of closed fibers in Subsections 3.2 and 3.3. Subsequently, Sections 4, 5 and 6 are devoted to proving 1, 2 and 3 of Theorem 1.1, respectively.

Acknowledgements. – I would like to express my thanks to my PhD advisor Jérémy Blanc for introducing me to this topic and discussing it with me. Moreover, I am very grateful to Marat Gizatullin for his thought-provoking comments and pointing out some carelessness in the first version of this text. Then, my thanks go to Adrien Dubouloz and Johannes Huisman for pointing out the moduli interpretation. Lastly, I am indebted to the referees whose reports helped to drastically simplify the exposition and allow the paper to be self-contained.

2. Preliminaries

2.1. Underlying definitions and results

For any $\mathbb{R}$-scheme X_0, denote by $(X_0)_{\mathbb{C}} = X_0 \times_{\mathbb{R}} \mathrm{Spec}(\mathbb{C})$ the *complexification* of X_0, where the morphism of schemes $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ is induced by the Galois extension $\mathbb{C}/\mathbb{R}$. Similarly, if a morphism $\psi\colon X_0 \to Y_0$ of $\mathbb{R}$-schemes is given, then we denote by $\psi_{\mathbb{C}}\colon (X_0)_{\mathbb{C}} \to (Y_0)_{\mathbb{C}}$ the base change from $\mathbb{R}$ to $\mathbb{C}$, and refer to it as the *complexification* of ψ.

A *real form* of a complex scheme X is a real scheme X_0 together with a $\mathbb{C}$-isomorphism $(X_0)_{\mathbb{C}} \xrightarrow{\sim} X$. Instead of real forms, one may study real structures, as the two are closely linked. A *real structure* on a complex scheme X is an anti-regular involution $\rho : X \to X$; by anti-regularity we mean that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\rho\quad} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{z \mapsto \overline{z}} & \mathrm{Spec}(\mathbb{C}) \end{array}$$

commutes. Given two real structures ρ on X and ρ' on X', a morphism $\theta\colon X \to X'$ is called *real* if $\theta \circ \rho = \rho' \circ \theta$, and we write $\theta\colon (X, \rho) \to (X', \rho')$. The real structures ρ on X and ρ' on X' are *equivalent* if there exists an isomorphism $\psi\colon (X, \rho) \xrightarrow{\sim} (X', \rho')$.

REMARK 2.1. – By the universal property of the blow-up, whenever we blow up a $\mathbb{C}$-scheme Y with real structure ρ' in a closed subscheme invariant under ρ', then there exists a real structure ρ on the blow-up $\pi\colon X \to Y$ such that π is real, see for example [22, II.7.14].