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DEGENERACY OF HOLOMORPHIC MAPS VIA ORBIFOLDS

by Erwan Rousseau

Abstract. — We use orbifold structures to deduce degeneracy statements for holomorphic maps into logarithmic surfaces. We improve former results in the smooth case and generalize them to singular pairs. In particular, we give applications on nodal surfaces and complements of singular plane curves.

Résumé (*Dégénérescence des applications holomorphes par le biais des orbifoldes*)
Nous utilisons les structures orbifoldes pour obtenir des résultats de dégénérescence des applications holomorphes dans les surfaces logarithmiques. Nous améliorons certains résultats déjà obtenus dans le cas lisse et les généralisons aux paires singulières. En particulier, nous illustrons nos résultats sur les surfaces nodales et les complémentaires de courbes planes singulières.

1. Introduction

It is now classical that the properties of holomorphic maps in compact complex manifolds are closely related to the properties of the canonical line bundle. More precisely, one can expect following Green-Griffiths [18] that the following is true:

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Conjecture 1.1. — *Let X be a projective manifold of general type, i.e. its canonical line bundle K_X is big. Then there exists a proper subvariety $Y \subsetneq X$ which contains every non-constant entire curve $f : \mathbb{C} \to X$.*

It can be observed that positivity properties of the canonical bundle can be generalized to a more general situation than the usual compact setting, and still give properties of degeneracy for holomorphic maps. For example, a classical result of Nevanlinna is

Theorem 1.2 ([30]). — *Let $a_1, \dots, a_k \in \mathbb{P}^1$ and $m_1, \dots, m_k \in \mathbf{N} \cup \infty$. If*

$$\sum_{i=1}^{k} \left(1 - \frac{1}{m_i}\right) > 2,$$

then every entire curve $f : \mathbb{C} \to \mathbb{P}^1$ which is ramified over a_i with multiplicity at least m_i is constant.

Following Green-Griffiths' philosophy, this degeneracy property should correspond to the positivity property of some canonical line bundle. Here one easily observes that the right canonical line bundle to consider is

$$K_{\mathbb{P}^1} + \sum_{i=1}^{k} \left(1 - \frac{1}{m_i}\right) a_i,$$

which can be seen as the canonical line bundle of the pair $(\mathbb{P}^1, \Delta)$ where $\Delta = \sum_{i=1}^{k} \left(1 - \frac{1}{m_i}\right) a_i$.

More generally, following Campana [6], a pair (X, Δ), consisting of a complex manifold X and a $\mathbb{Q}$-divisor $\Delta = \sum_{i=1}^{k} \left(1 - \frac{1}{m_i}\right) Z_i$, is called a geometric orbifold. Positivity properties of the orbifold canonical line bundle, $K_X + \Delta$, should provide degeneracy statements for orbifold entire curves $f : \mathbb{C} \to X$, i.e. entire curves with ramification (see section 3 below for precise definitions).

In this paper we shall study the case of surfaces improving and generalizing results of a previous work [31]. The point of view we adopt here consists in working with the different notions of *orbifolds* that have appeared in the literature: the V-manifolds of Satake, the orbifolds of Thurston, the algebraic stacks of Grothendieck, Deligne and Mumford, and the geometric orbifolds of Campana.

In particular, as initiated in [8], we extend to the orbifold setting the strategy of Bogomolov [3] which uses symmetric differentials to obtain hyperbolicity properties for surfaces which satisfy $c_1^2 - c_2 > 0$. More precisely, we use Kawasaki-Toën's Riemann-Roch formula on stacks ([19], [34]) to produce orbifold symmetric differentials.

Then, in the case of smooth (geometric) orbifolds (i.e. X is smooth and $\lceil\Delta\rceil$, the support of Δ, is a normal crossing divisor), we obtain using moreover McQuillan's techniques [25] as in [31]:

THEOREM A. — *Let (X, Δ) be a smooth projective orbifold surface of general type, i.e $K_X + \Delta$ is big, $\Delta = \sum_i (1 - \frac{1}{m_i}) C_i$. Denote $g_i := g(C_i)$ the genus of the curve C_i and $\bar{c}_1, \bar{c}_2$ the logarithmic Chern classes of $(X, \lceil\Delta\rceil)$. If*

$$\bar{c}_1^2 - \bar{c}_2 - \sum_{i=1}^{n} \frac{1}{m_i}(2g_i - 2 + \sum_{j\neq i} C_i C_j) + \sum_{1\leq i\leq j\leq n} \frac{C_i C_j}{m_i m_j} > 0, \tag{1.1}$$

then there exists a proper subvariety $Y \subsetneq X$ such that every non-constant entire curve $f : \mathbb{C} \to X$ which is an orbifold morphism, i.e ramified over C_i with multiplicity at least m_i, verifies $f(\mathbb{C}) \subset Y$.

One advantage of this new approach is that we can generalize it to the singular case, for example when the orbifold surface (X, Δ) is Kawamata log terminal, following the terminology of the Mori Program (see for example [23]). This point of view unifies several former results (e.g. [12], [17] and [5]) where people have noticed that singularities can help to prove degeneracy statements on holomorphic maps. The key point here is to realize that singularities help to produce orbifold symmetric differentials on the stack associated to the orbifold.

As applications, we obtain as a first example (compare with [12] and [17]):

THEOREM B. — *Let $C \subset \mathbb{P}^2$ be a curve of degree $d \geq 4$ with n nodes and c cusps. If*

$$-d^2 - 15d + \frac{75}{2} + \frac{1079}{96}c + 6n > 0,$$

then there exists a curve $D \subset \mathbb{P}^2$ which contains any non-constant entire curve $f : \mathbb{C} \to \mathbb{P}^2 \setminus C$.

The above numerical conditions should be seen as the equivalent of

$$c_1^2 - c_2 > 0$$

in the orbifold setting. A second example is the case of nodal surfaces $X \subset \mathbb{P}^3$ of general type of degree d with l nodes where we recover a result of [5] giving the existence of orbifold symmetric differentials as soon as $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$, which is unfortunately not satisfied for $d = 5$ where the maximum number of nodes is 31.

We extend our study to higher order orbifold jet differentials and obtain, towards the existence of an hyperbolic quintic,

THEOREM C. — *Let $X \subset \mathbb{P}^3$ be a nodal quintic with the maximum number of nodes,* 31. *Then every classical orbifold entire curve satisfies an algebraic differential equation of order* 3.

The paper is organized as follows. In section 2, we recall the basic facts on orbifold structures. In section 3, we describe orbifold symmetric differentials and orbifold morphisms. Then, in section 4, we study some properties of holomorphic disks in orbifolds. In section 5, we recall Kawasaki-Toën's Riemann-Roch formula on orbifolds. In section 6, we study the smooth case and in section 7, the singular case. In section 8, we give applications to complements of plane curves and nodal surfaces. Finally, in section 9, we give definitions and applications of orbifold jet differentials.

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2. Orbifolds as pairs

As in [16] (or [13], §14) we look at orbifolds as a particular type of log pairs. (X, Δ) is a log pair if X is a normal algebraic variety (or a normal complex space) and $\Delta = \sum_i d_i D_i$ is an effective $\mathbb{Q}$-divisor where the D_i are distinct, irreducible divisors and $d_i \in \mathbb{Q}$.

For orbifolds, we need to consider only pairs (X, Δ) such that Δ has the form

$$\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) D_i,$$

where the D_i are prime divisors and $m_i \in \mathbb{N}$. These pairs are called *geometric orbifolds* by Campana in [6] and [7].

Definition 2.1. — *An* orbifold chart *on X compatible with Δ is a Galois covering $\varphi : U \to \varphi(U) \subset X$ such that*

1. *U is a domain in $\mathbb{C}^n$ and $\varphi(U)$ is open in X,*
2. *the branch locus of φ is $\lceil \Delta \rceil \cap \varphi(U)$,*
3. *for any $x \in U'' := U \setminus \varphi^{-1}(X_{sing} \cup \Delta_{sing})$ such that $\varphi(x) \in D_i$, the ramification order of φ at x verifies $ord_{\varphi}(x) = m_i$.*

Definition 2.2. — *An orbifold $\mathscr{X}$ is a log pair (X, Δ) such that X is covered by orbifold charts compatible with Δ.*