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SELF-IMPROVING BOUNDS FOR THE NAVIER-STOKES EQUATIONS

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ABSTRACT. — We consider regular solutions to the Navier-Stokes equation and provide an extension to the Escauriaza-Seregin-Sverak blow-up criterion in the negative regularity Besov scale, with regularity arbitrarly close to -1. Our results rely on turning a priori bounds for the solution in negative Besov spaces into bounds in the positive regularity scale.

RÉSUMÉ (*Estimations de bootstrap* a priori *pour Navier-Stokes*). — On considère des solutions régulières des équations de Navier-Stokes pour lesquelles on prouve une extension du critère d'explosion d'Escauriaza-Seregin-Sverak dans l'échelle des espaces de Besov de régularité négative, arbitrairement proche de -1. Nos résultats reposent sur l'amélioration d'estimations *a priori* en régularité négative pour devenir à régularité positive.

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1. Introduction

We consider the incompressible Navier-Stokes equations in $\mathbb{R}^3$,

$$\text{(NS)} \qquad \begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \otimes u) - \nabla \pi, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0 \end{cases}$$

for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$, where $u = (u_i(x, t))_{i=1}^3 \in \mathbb{R}^3$ is the velocity vector field, $\pi(x, t) \in \mathbb{R}$ is the associated pressure function and

$$\nabla \cdot (u \otimes u) := \Big(\sum_{j=1}^{d} \partial_{x_j} (u_i u_j) \Big)_{i=1}^{d}.$$

In the pioneering work [11], J. Leray proved the existence of global turbulent (weak in the modern terminology) solutions of (NS) for initial data with finite kinetic energy, i.e. initial data in L^2. These solutions need not be unique or preserve regularity of the initial data. In this same work, J. Leray proved that for regular enough initial data (namely H^1 initial data), a local (in time) unique solution exists. He also proved that as long as this solution is regular enough, it is unique among all the possible turbulent solutions, and moreover, if such a turbulent solution satisfies

$$u \in L^p([0, T[; L^q(\mathbb{R}^3)) \text{ with } \frac{2}{p} + \frac{3}{q} = 1, \; q > 3, \tag{1.1}$$

then the solution remains regular on $[0, T]$ and can be extended beyond time T. This is now known as Serrin's criterion.

On the other hand, there is a long line of works on constructing local in time solutions, from H. Fujita and T. Kato (see [9]) to H. Koch and D. Tataru (see [10]). For these results, the main feature is that the initial data belongs to spaces which are invariant under the scaling of the equations. Between [9] and [10], T. Kato (see [8]) proved wellposedness of (NS) for initial data u_0 in L^3. In this framework of local in time (strong, e.g. unique) solutions, Serrin's criterion may be understood as a non blow-up criterion at time T: e.g. if u is a strong solution with $u_0 \in L^3(\mathbb{R}^3)$, that is $u \in C([0, T[; L^3(\mathbb{R}^3))$, and if (1.1) is satisfied, then one may (continuously and uniquely) extend the solution u past time T.

In the recent important work [7], L. Escauriaza, G. Seregin and V. Šverák obtained the endpoint version of Serrin's criterion, using blow-up techniques to construct a special solution vanishing at blow-up time and then backward uniqueness to rule out its existence. Earlier work of Giga and Von Wahl proved this endpoint under a continuity in time assumption in L^3, and such a continuity result was recently improved to match the local in time theory by Cheskidov-Shvydkoy [4].

Our first theorem (Theorem 1 below) may be seen as an extension of the end-point criterion by Escauriaza-Seregin-Šverák, in the negative regularity scale. Before providing an exact statement, we need to introduce a few notations and definitions.

Since we are interested in smooth (or at least strong in the Kato sense) solutions, (NS) is equivalent for our purpose with its integral formulation, where the pressure has been disposed of with the projection operator $\mathbb{P}$ over divergence free vector fields:

$$u = S(t)u_0 - \int_0^t \mathbb{P}S(t-s)\nabla\cdot(u\otimes u)(s)\,ds = u_L + B(u,u) \tag{1.2}$$

where $S(t) = \exp(t\mathbb{P}\Delta) = \mathbb{P}\exp(t\Delta)$ is the Stokes flow (which is nothing but the heat flow in $\mathbb{R}^3$ on divergence free vector fields) and $B(u,u)$ is the Duhamel term which reads, component wise

$$B(f,g) = -\int_0^t R_jR_kR_l|\nabla|S(t-s)(fg)(s)\,ds, \tag{1.3}$$

where the $R_{(\cdot)}$ are the usual Riesz transforms (recall $\mathbb{P}$ is a Fourier multiplier with matrix valued symbol $\mathrm{Id}-|\xi|^{-2}\xi\otimes\xi$). We will denote the Lebesgue norm by

$$\|f\|_p = \|f\|_{L^p} = \left(\int_{\mathbb{R}^3}|f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

Let us recall a definition of Besov spaces using the heat flow $S(\sigma)$.

Definition 1.1. — *Let $Q(\sigma) = \sigma\partial_\sigma S(\sigma)$. We define $\dot B^{s,q}_p$ as the set of tempered distributions f such that*

- *the integral $\int_{1/N}^N Q(\sigma)f\,d\sigma/\sigma$ converges to f when $N\to+\infty$ as a tempered distribution if $s<\frac{d}{p}$ and after taking the quotient with polynomials if not, and*
- *the function $\sigma^{-s/2}\|Q(\sigma)f\|_p$ is in $L^q(d\sigma/\sigma)$; its norm defines the Besov norm of f:*

$$\|f\|^q_{\dot B^{s,q}_p} = \int_0^{+\infty}\sigma^{-sq/2}\|Q(\sigma)f\|^q_p\,\frac{d\sigma}{\sigma}. \tag{1.4}$$

We recall that the usual (homogeneous) Sobolev spaces $\dot H^s$, defined through the Fourier transform by $|\xi|^s\widehat{f}(\xi)\in L^2$, may be identified with $\dot B^{s,2}_2$, while the critical Sobolev embedding holds as follows: $\dot B^{s,q}_p \hookrightarrow \dot B^{\rho,\lambda}_r$ provided $s-d/p = \rho-d/r$, $s\geq\rho$ and $q\leq\lambda$, as well as $\dot B^{s,q}_p\hookrightarrow L^r_x$ provided $s-d/p=-d/r$, $s\geq 0$ and $q\leq r$.

We are now in a position to state our first result:

THEOREM 1. — *Let u be a local in time solution to (NS) such that $u_0 \in \dot{H}^{1/2}$. Assume that there exist $p \in]3, +\infty[$ and $q < 2p'$ such that*

$$\sup_{t\in[0,T[} \|u(\cdot,t)\|_{\dot{B}_p^{3/p-1,q}} < +\infty, \tag{1.5}$$

then the solution may be uniquely extended past time T.

We remark that our hypothesis allows for smooth, compactly supported data; actually, one may simply assume that the vorticity $\omega_0 = \nabla \wedge u_0$ belongs to $L^{3/2}$. By Sobolev embedding and the Biot-Savart law, this implies that u_0 belongs to $\dot{H}^{\frac{1}{2}} \subset \dot{B}_p^{3/p-1,2}$. Hence by local Cauchy theory so does u and (1.5) is finite at least for small times.

It is of independent interest to consider the case of L^3 data, without any extra regularity hypothesis:

THEOREM 2. — *Let u be a local in time strong solution to (NS) with data u_0 in $L^3 \cap \dot{B}_p^{3/p-1,q}$, with $3 < p < +\infty$ and $q < 2p'$. Assume that*

$$\sup_{t\in[0,T[} \|u(\cdot,t)\|_{\dot{B}_p^{3/p-1,q}} < +\infty, \tag{1.6}$$

then the solution may be uniquely extended past time T.

The restriction on q for the data implies that $q < 3$ as $p > 3$. As such, our result does not include the L^3 case, as we are still assuming a subtle decay hypothesis through the q indice. However, the restriction is mostly technical and all is required to lift it is to generalize the results from [6], most specifically the compactness result which is only stated in L^3 rather than in the Besov scale. This will be adressed elsewhere, providing generalizations of the present note and the results of [6]. Our purpose here is to illustrate that these blow up criterions do not require positive regularity on the data; in fact, they will extend to non L^3 data into the negative Besov scale.

Both Theorem 1 and 2 rely crucially on improving the rather weak a priori bound on u from the hypothesis. Such "self-improvements" are of independent interest and we state examples of them below. We start with a (spatial) regularity improvement for negative Besov-valued data (see the forthcoming Remark 2.7 on the p range restriction which is only technical).

THEOREM 3. — *Let u be a local in time strong solution to (NS) with data $u_0 \in \dot{B}_p^{3/p-1,q}$, with $3 < p < 6$ and $q < +\infty$. Assume that*

$$\sup_{t\in[0,T[} \|u(\cdot,t)\|_{\dot{B}_p^{3/p-1,\infty}} \leq M, \tag{1.7}$$