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PROPAGATION OF SINGULARITIES AROUND A LAGRANGIAN SUBMANIFOLD OF RADIAL POINTS

BY NICK HABER & ANDRÁS VASY

ABSTRACT. — In this work we study the wavefront set of a solution u to $Pu = f$, where P is a pseudodifferential operator on a manifold with real-valued homogeneous principal symbol p, when the Hamilton vector field corresponding to p is radial on a Lagrangian submanifold Λ contained in the characteristic set of P. The standard propagation of singularities theorem of Duistermaat-Hörmander gives no information at Λ. By adapting the standard positive-commutator estimate proof of this theorem, we are able to conclude additional regularity at a point q in this radial set, assuming some regularity around this point. That is, the a priori assumption is either a weaker regularity assumption at q, or a regularity assumption near but not at q. Earlier results of Melrose and Vasy give a more global version of such analysis. Given some regularity assumptions around the Lagrangian submanifold, they obtain some regularity at the Lagrangian submanifold. This paper microlocalizes these results, assuming and concluding regularity only at a particular point of interest. We then proceed to prove an analogous result, useful in scattering theory, followed by analogous results in the context of Lagrangian regularity.

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Résumé (*Propagation des singularités près d'une sous-variété Lagrangienne des points radiaux*)

Dans cet article on étudie le spectre singulier, $\mathrm{WF}(u)$, pour les solutions de l'équation $Pu = f$, où P est un operateur pseudo-différentiel sur une variété de la classe C^∞, X, avec symbole principal homogène p, si le champ Hamiltonien de p est radial sur une sous-variété lagrangienne, Λ, contenue dans l'ensemble caractéristique de P. Le théorème classique de Duistermaat et Hörmander ne fournit aucune information sur Λ. Nous adaptons la preuve de ce théorème utilisant des commutateurs positifs, et prouvons que la solution possède d'une régularité additionelle près d'un point q si on suppose certaine régularité au fond. C'est à dire, l'hypothèse a priori est soit une hypothèse de régularité plus faible à q, soit une hypothèse de régularité près de, mais pas à q. Les résultats plus anciens de Melrose et Vasy donnent une version plus globale de cette analyse. Cet article fournit une version microlocale des résultats de ces auteurs; on suppose et prouve la régularité seulement près du point d'intérêt, q. Nous prouvons aussi un résultat similaire qui est utile dans la théorie de la diffusion, et aussi des résultats de la régularité lagrangienne.

1. Introduction

This paper studies the wavefront set of a solution u to $Pu = f$, where P is a pseudodifferential operator on a manifold with real-valued homogeneous principal symbol p, when the Hamilton vector field corresponding to p is radial on a Lagrangian submanifold contained in the characteristic set of P. According to a theorem of Duistermaat-Hörmander, see [3], singularities propagate along bicharacteristics of this Hamilton vector field. This theorem gives us no information about the wavefront set when the Hamilton vector field is radial. Melrose in [13] and Vasy in [16] gave a global analysis of the propagation of singularities around a Lagrangian submanifold of radial points. By adapting the standard positive commutator estimate proof of this theorem, we microlocalize these results. (This had been done in a special case by Vasy in [15].)

After proving such a result, we proceed to prove an analogous result, useful in scattering theory, in particular in resolvent estimates. Analogous to the standard propagation of singularities, microlocal Sobolev bounds on u_τ which are uniform in $\tau \in [0, 1]$ or $(0, 1]$ propagate forward along bicharacteristics, assuming uniform Sobolev bounds for $(P - i\tau)u$, where now P is of order 0 (see, for instance, [13]). We prove a corresponding statement around a Lagrangian submanifold of radial points, generalizing to solutions of $P - iQ_\tau$, with P, Q_τ of equal order (not necessarily 0), with suitable boundedness and positivity assumptions on Q_τ. This is again a microlocal result which generalizes a global result given in [13].

Lastly, we prove analogs in the context of Lagrangian regularity, essentially replacing "u is microlocally $H^s(X)$" with "u is microlocally a Lagrangian distribution". This follows the analyses of Hassell, Melrose and Vasy in [5] and [6].

It should be emphasized that these results are completely local. That is, in order to conclude regularity for u at a point q in this Lagrangian submanifold, we need only have regularity for f in an arbitrarily small neighborhood of q. At times we also need regularity assumptions on u around the bicharacteristics approaching the smallest conic subset containing the $\mathbb{R}_+$-orbit containing q, and at other times we also need a priori regularity assumptions on u-–it is important to note that these requirements are again local around q. Thus we do not, for instance, require regularity assumptions around the whole Lagrangian submanifold.

Under the nondegeneracy assumption $dp \neq 0$, the largest-dimensional subspace on which a Hamilton vector field can be radial is a Lagrangian submanifold. This occurs naturally in many applications, including geometric scattering theory. Indeed, these results generalize a series of results in [13]. For the treatment of the opposite extreme, that is, that of an isolated radial point, see for instance the paper of Guillemin and Schaeffer [4], as well as the above mentioned papers of Hassell, Melrose and Vasy [5, 6]. The works of Herbst and Skibsted [7, 8] also study cases of this last scenario, while Bony, Fujiié, Ramond and Zerzeri in [1] study a semiclassical version of this last scenario.

In Section 1.1, we introduce basic microlocal terminology. We then state the standard (principal-type) propagation of singularities theorems and discuss radial points in Section 1.2. In Section 1.3, we discuss the cosphere bundle as a quotient of the cotangent bundle (excluding the zero section). As it is at times easier to discuss dynamics on the cosphere bundle than it is on the cotangent bundle, we regard certain conic sets, such as wavefront sets, to be subsets of the cosphere bundle. We then state the main theorems of the paper in Section 1.4, leaving out the more technical statements of Theorems 6.3 and 6.4 and instead giving Theorem 1.7, a simplified version. In Section 1.5, we sketch the proofs of these theorems. The theorems contain 'threshold' values (s_0, s_1) that have explicit values which are complicated to state in generality but can be refined considerably under additional assumptions. We thus delay discussing these values until Section 1.5.1. We then proceed to prove Theorem 1.5 in Sections 2, 3, and 4. Theorem 1.4 follows as a special case. In Section 2, we analyze the Hamiltonian dynamics around the radial points. In Section 3, we give the positive commutator proof of Theorem 1.5, assuming the existence of certain operators. In Section 4, we construct these operators. In Section 5, we adapt these constructions for Theorem 1.6. In Section 6, we review the notion from [5] of iterative regularity, in the context of Lagrangian regularity, state and prove Theorems 6.3 and 6.4.