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COHOMOLOGICAL CONDITIONS ON ENDOMORPHISMS OF PROJECTIVE VARIETIES

by Holly Krieger & Paul Reschke

Abstract. — We characterize possible periodic subvarieties for surjective endomorphisms of complex abelian varieties in terms of the eigenvalues of the cohomological actions induced by the endomorphisms, extending previous work in this direction by Pink and Roessler [20]. By applying our characterization to induced endomorphisms of Albanese varieties, we draw conclusions about the dynamics of surjective endomorphisms for a broad class of projective varieties. We also analyze several classes of surjective endomorphisms that are distinguished by properties of their cohomological actions.

Résumé (*Conditions cohomologiques sur les endomorphismes des variétés projectives*). — Pour des endomorphismes surjectives sur des variétés abéliennes complexes, nous décrivons les sous-variétés périodiques quit peuvent se présenter au moyens des valeurs propres des actions cohomologiques des endomorphismes. Cette entreprise élargit quelques idées des Pink et Roessler [20]. Nous utilisons ensuite la description des sous-variétés périodiques en étudiant des endomorphismes sur des variétés d'Albanese qui viennent des endomorphismes sur des sous-variétés. Nous aussi définissons et étudions quelques catégories des endomorphismes qui se différencient par certaines propriétés de leurs actions cohomologiques.

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1. Introduction

In this note, we study the nature of periodic subvarieties for endomorphisms of smooth complex projective varieties. The starting point for our investigation is a theorem due to Pink and Roessler:

Theorem 1.1 ([20], Theorem 2.4). — *Let $f : A \to A$ be an isogeny of a complex abelian variety A, and suppose that no eigenvalue of $f^*|_{H^{1,0}(A)}$ is a root of unity. Suppose that $V \subseteq A$ is a reduced and irreducible subvariety satisfying $f(V) = V$. Then V is a translate of an abelian subvariety of A.*

By the Lefschetz Fixed-Point Theorem, the eigenvalue condition in Theorem 1.1 guarantees that f has a fixed point, and therefore is conjugate by a translation to an isogeny, even if f is only assumed to be a surjective endomorphism (i.e., not necessarily a homomorphism). Thus the conclusion holds for any surjective endomorphism f satisfying the eigenvalue condition. (See §2.3 and §2.4 below.) We will not assume in the following that a surjective endomorphism of an abelian variety is an isogeny.

We extend Theorem 1.1 to the case where f^* may have eigenvalues on $H^{1,0}(A)$ that are roots of unity; here, $\kappa(V)$ denotes the Kodaira dimension of any smooth birational model of a variety V:

Theorem 1.2. — *Let f be a surjective endomorphism of a complex abelian variety A, and suppose that $V \subseteq A$ is a reduced and irreducible subvariety satisfying $f(V) = V$. Then there is a reduced and irreducible subvariety $W \subseteq V$ with $\kappa(W) = \dim(W) = \kappa(V)$, and some iterate f^k, such that $V = \mathrm{Stab}^0_A(V) + W$ and $f^k(\mathrm{Stab}^0_A(V) + w) = \mathrm{Stab}^0_A(V) + w$ for every $w \in W$.*

The proof of Theorem 1.2 has a similar flavor to the original proof of Theorem 1.1 by Pink and Roessler, and also echoes some of the content from their Theorem 3.1 in [21]: by Ueno [26], all subvarieties of A can be built from tori and varieties of general type; we then apply Kobayashi and Ochiai [14], which states that every rational self-map of a variety of general type has finite order. (See §2.1 and §2.2 below.) Note that W may be singular or zero-dimensional; in particular, if $\kappa(V) = 0$, then V is a translate of an abelian subvariety of A.

As a corollary of Theorem 1.2, we recover the following mild strengthening of the theorem of Pink and Roessler:

Corollary 1.3. — *Let f be a surjective endomorphism of a complex abelian variety A, and suppose that $V \subseteq A$ is a reduced and irreducible subvariety satisfying $f(V) = V$. Let u_f denote the number of root-of-unity eigenvalues of $f^*|_{H^{1,0}(A)}$ with multiplicity. Then*

$$\kappa(V) \leq u_f;$$

in fact, the inequality is strict except possibly if $\kappa(V) = 0$.

Suppose now that X is an arbitrary smooth complex projective variety, and that f is a surjective endomorphism of X. Since the Albanese variety $\mathrm{Alb}(X)$ is generated by the image of X under the Albanese map a_X, f induces a surjective endomorphism F of $\mathrm{Alb}(X)$; moreover, if $a_X(X) \neq \mathrm{Alb}(X)$, then $a_X(X)$ is a reduced and irreducible proper subvariety of $\mathrm{Alb}(X)$ satisfying $F(a_X(X)) = a_X(X)$. (See §3.1 below.) So we can use Theorem 1.2 to draw conclusions about endomorphisms of varieties with non-surjective Albanese maps, as in:

COROLLARY 1.4. — *Let X be a smooth complex projective variety with $a_X(X) \neq \mathrm{Alb}(X)$, and suppose that f is an infinite-order surjective endomorphism of X. Then some iterate f^k preserves a non-trivial fibration on X. In particular, X contains proper positive-dimensional subvarieties which are periodic for f.*

Note that any variety X in Corollary 1.4 must have $\kappa(a_X(X)) > 0$; but it will follow from the proof that this is not necessarily true for the periodic subvariety, so this corollary cannot be used for induction. Note also that a smooth curve with a non-surjective Albanese map is necessarily hyperbolic and hence, by the De Franchis Theorem, does not admit any infinite-order endomorphisms. If we write $a_X(X) = B + W$, where B is the stabilizer of $a_X(X)$ in $\mathrm{Alb}(X)$ and W has $\kappa(W) = \dim(W)$, then the fibers in Corollary 1.4 are the pre-images under a_X of $B + w$ for all $w \in W$. (See §3.2 below.)

Corollary 1.4 relates to two recently proposed conjectures regarding Zariski dense orbits of points under iterated maps. Reichstein, Rogalski, and Zhang [23] conjectured that a wild automorphism—an automorphism for which every orbit is Zariski—can only arise on an abelian variety; Corollary 1.4 gives an alternate proof of the result in Proposition 5.1(a) in [23] that a variety whose Albanese map is non-surjective cannot admit a wild automorphism, and in fact shows that such a variety cannot admit a wild endomorphism. Medvedev and Scanlon [15] conjectured that an endomorphism with no periodic non-trivial fibration always has at least one Zariski dense orbit; Corollary 1.4 shows that the scope of this conjecture can be restricted to varieties whose Albanese maps are surjective.

We turn now to an assessment of certain classes of endomorphisms that are characterized by cohomological properties.

DEFINITION 1.5 ([17],[28]). — *Let f be a surjective endomorphism of a projective variety X. We say that f is **polarized** if there is an ample line bundle $L \in \mathrm{Pic}(X)$ such that $f^*(L) = L^{\otimes q}$ for some integer $q > 1$.*

The study of polarized endomorphisms—and those varieties which carry them—is of particular interest to dynamicists. For an endomorphism f of a complex variety X, Fakhruddin [6] showed that the condition that f is polarized is equivalent to the existence of an embedding $i : X \to \mathbb{P}^N$ and a morphism