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BLOW-UP OF THE CRITICAL SOBOLEV NORM FOR NONSCATTERING RADIAL SOLUTIONS OF SUPERCRITICAL WAVE EQUATIONS ON $\mathbb{R}^3$

by Thomas Duyckaerts & Tristan Roy

Abstract. — We consider the wave equation in space dimension 3, with an energy-supercritical nonlinearity which can be either focusing or defocusing. For any radial solution of the equation, with positive maximal time of existence T, we prove that one of the following holds: (i) the norm of the solution in the critical Sobolev space goes to infinity as t goes to T, or (ii) T is infinite and the solution scatters to a linear solution forward in time. We use a variant of the channel of energy method, relying on a generalized L^p-energy which is almost conserved by the flow of the radial linear wave equation.

Résumé (*Explosion d'une norme de Sobolev critique pour les solutions radiales non-dispersives de l'équation des ondes surcritique sur* $\mathbb{R}^3$). — Considérons l'équation des ondes avec une non-linéarité surcritique pour l'énergie, focalisante ou défocalisante, en dimension 3 d'espace. On démontre que toute solution radiale de l'équation, avec un temps d'existence maximal T, vérifie une des deux propriétés suivantes : (i) la norme de la solution dans l'espace de Sobolev critique tend vers l'infini quand t tend vers T ; (ii) T est infini, et la solution est asymptotiquement proche d'une solution linéaire pour des temps infiniment grands. La démonstration utilise une variante de la méthode des canaux d'énergie basée sur une énergie généralisée (définie dans un espace L^p à poids) qui est presque conservée par le flot de l'équation des ondes linéaires.

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1. Introduction

In many recent works, global existence and scattering for solutions of a supercritical dispersive equation were proved assuming that an appropriate critical norm remains bounded close to the maximal time of existence. The goal of this article is to prove, in a specific case, a slightly stronger result, namely that it is sufficient to assume the boundedness of the critical norm only along a sequence of times going to the time of existence. More precisely, we consider the supercritical wave equation in space dimension 3:

$$\begin{cases} \partial_{tt}w - \triangle w = \iota |w|^{p-1}w, & (t,x) \in I \times \mathbb{R}^3, \\ u(0,x) := u_0(x) \\ \partial_t u(0,x) := u_1(x) \end{cases} \tag{1.1}$$

with real-valued initial data

$$w(0,x) := w_0(x) \in \dot{H}^{s_c}(\mathbb{R}^3), \quad \partial_t w(0,x) := w_1(x) \in \dot{H}^{s_c-1}(\mathbb{R}^3), \tag{1.2}$$

where $(t,x) \in I \times \mathbb{R}^3$, I is an interval such that $0 \in I \subset \mathbb{R}$, $p > 5$, $\iota = +1$ (focusing case) or $\iota = -1$ (defocusing case), s_c is the critical Sobolev exponent, i.e., $s_c = \frac{3}{2} - \frac{2}{p-1}$, and $\dot{H}^{s_c}(\mathbb{R}^3)$ is the usual homogeneous Sobolev space.

Equation (1.1) is locally well-posed in $\dot{H}^{s_c}(\mathbb{R}^3) \times \dot{H}^{s_c-1}(\mathbb{R}^3)$: for any initial data (w_0, w_1), there exists a solution

$$\vec{w} := (w, \partial_t w) \in C^0\big((T_-(w), T_+(w)), \dot{H}^{s_c}(\mathbb{R}^3) \times \dot{H}^{s_c-1}(\mathbb{R}^3)\big)$$

defined on a maximal interval of existence $(T_-(w), T_+(w))$ that satisfies (1.1), (1.2) in the Duhamel sense, and is unique in a natural class of functions. It has the following scaling invariance: if $\lambda > 0$ and w is a solution, then w_λ, defined by

$$w_\lambda(t,x) := \lambda^{\frac{2}{p-1}} w(\lambda t, \lambda x)$$

is also a solution of (1.1), with maximal interval of existence $(\lambda^{-1}T_-(w), \lambda^{-1}T_+(w))$, that satisfies

$$\|\vec{w}_\lambda(0)\|_{\dot{H}^{s_c}(\mathbb{R}^3)\times\dot{H}^{s_c-1}(\mathbb{R}^3)} = \|(w_0,w_1)\|_{\dot{H}^{s_c}(\mathbb{R}^3)\times\dot{H}^{s_c-1}(\mathbb{R}^3)}\,. \tag{1.3}$$

With the additional assumption $(w_0, w_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, the energy

$$E(\vec{w}(t)) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w(t,x)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |\partial_t w(t,x)|^2\,dx - \iota\frac{1}{p+1}\int_{\mathbb{R}^3} |w(t,x)|^{p+1}\,dx$$

is well-defined for all t and independent of time. The assumption $p > 5$ is equivalent to $s_c > 1$: the equation is energy-supercritical, and the energy has little utility in the study of global well-posedness or related properties.

Our goal is to classify the solutions of (1.1) according to their dynamics. The solution w is said to *scatter* forward in time if $T_+(w) = +\infty$ and if there exists a solution w_{L} of the linear wave equation

$$\partial_{tt}w_{\mathrm{L}} - \triangle w_{\mathrm{L}} = 0 \tag{1.4}$$

such that

$$\lim_{t\to+\infty} \|\vec{w}_{\mathrm{L}}(t) - \vec{w}(t)\|_{\dot{H}^{s_c}(\mathbb{R}^3)\times\dot{H}^{s_c-1}(\mathbb{R}^3)} = 0. \tag{1.5}$$

It was proved in [19] in the defocusing case that radial solutions of (1.1) that are bounded in the critical space scatter. More precisely, for any solution of (1.1), (1.2) with $\iota = -1$, and radial initial data (w_0, w_1)

$$\limsup_{t\to T_+(w)} \|\vec{w}(t)\|_{\dot{H}^{s_c}(\mathbb{R}^3)\times\dot{H}^{s_c-1}(\mathbb{R}^3)} < \infty \Longrightarrow w \text{ scatters forward in time.} \tag{1.6}$$

This was later extended to the nonradial, defocusing case in [23], and the radial, focusing case in [10] (see also [20], [2], [4] in higher dimensions). Note that it follows from the scaling invariance of the equation and (1.3) that it is impossible to give a lower a priori bound of $T_+(w)$ only in terms of the $\dot{H}^{s_c}(\mathbb{R}^3)\times\dot{H}^{s_c-1}(\mathbb{R}^3)$-norm of (w_0, w_1): in particular, even the implication

$$\limsup_{t\to T_+(w)} \|\vec{w}(t)\|_{\dot{H}^{s_c}(\mathbb{R}^3)\times\dot{H}^{s_c-1}(\mathbb{R}^3)} < \infty \Longrightarrow T_+(w) = +\infty,$$

weaker than (1.6), does not follow from the local Cauchy theory for equation (1.1).

In this paper, we improve (1.6) in the radial case:

Theorem 1.1. — *Assume $p > 5$. Let w be a solution of* (1.1) *with radial data $(w_0, w_1) \in \dot{H}^{s_c}(\mathbb{R}^3)\times\dot{H}^{s_c-1}(\mathbb{R}^3)$. Then:*

- *either*

$$\lim_{t\to T_+(w)} \|(w(t), \partial_t w(t))\|_{\dot{H}^{s_c}(\mathbb{R}^3)\times\dot{H}^{s_c-1}(\mathbb{R}^3)} = +\infty \tag{1.7}$$

- *or $T_+(w) = +\infty$ and w scatters forward in time.*

An analogous statement holds for negative times.

Note that Theorem 1.1 is equivalent to (1.6), with the limit superior replaced by a limit inferior. We do not know any direct application of this qualitative improvement, however its analog in the case $p = 5$, $\iota = 1$ is crucial in the proof of the soliton resolution conjecture for the energy-critical wave equation in [9].

Theorem 1.1 means exactly that solutions of (1.1) are of one of the following three types: scattering solutions, solutions blowing-up in finite time with a critical norm going to infinity at the maximal time of existence, and global solutions with a critical norm going to infinity for infinite times.

In the defocusing case $\iota = -1$, it is conjectured that all solutions of (1.1) with initial data in the critical space $\dot{H}^{s_c}\times\dot{H}^{s_c-1}$ scatter. The difficulty of this conjecture is of course the lack of conservation law at the level of regularity of this critical space. The only supercritical dispersive equations for which