PANORAMAS ET SYNTHÈSES 24

ARC SPACES AND ADDITIVE INVARIANTS IN REAL ALGEBRAIC AND ANALYTIC GEOMETRY

Michel Coste
Toshizumi Fukui
Krzysztof Kurdyka
Clint McCrory
Adam Parusiński
Laurentiu Paunescu

Société Mathématique de France 2007
Publié avec le concours du Centre National de la Recherche Scientifique

Michel Coste
IRMAR (UMR CNRS 6625), Université de Rennes 1, Campus de Beaulieu,
35042 Rennes CEDEX (France).
E-mail : michel.coste@univ-rennes1.fr

Toshizumi Fukui
Department of Mathematics, Faculty of Science, Saitama University,
255 Shimo-Okubo, Urawa 338 (Japan).
E-mail : tfukui@rimath.saitama-u.ac.jp

Krzysztof Kurdyka
Laboratoire de Mathématiques, UMR 5175 du CNRS, Université de Savoie,
Campus Scientifique, 73 376 Le Bourget-du-Lac Cedex (France).
E-mail : kurdyka@univ-savoie.fr

Clint McCrory
Department of Mathematics, University of Georgia, Athens, GA 30602 (USA).
E-mail : clint@math.uga.edu

Adam Parusiński
Département de Mathématiques, UMR 6093 du CNRS,
Université d'Angers, 2, Bd Lavoisier, 49045 Angers Cedex 01 (France).
E-mail : adam.parusinski@univ-angers.fr

Laurentiu Paunescu
School of Mathematics and Statistics, University of Sydney, NSW 2006 (Australia).
E-mail : laurent@maths.usyd.edu.au

2000 *Mathematics Subject Classification.* — 14Pxx, 14P05, 14P10, 14P25, 32B20, 32C05, 58A07.

***Key words and phrases.* —** Real algebraic sets, semi-algebraic sets, arc spaces, arc-symmetric sets, arc-analytic mappings, blow-analytic equivalence, Euler integral, algebraically constructible functions, topological invariants.

ARC SPACES AND ADDITIVE INVARIANTS IN REAL ALGEBRAIC AND ANALYTIC GEOMETRY

Michel Coste, Toshizumi Fukui, Krzysztof Kurdyka, Clint McCrory, Adam Parusiński, Laurentiu Paunescu

***Abstract.* —** In this volume we present some new trends in real algebraic geometry based on the study of arc spaces and additive invariants of real algebraic sets. Generally, real algebraic geometry uses methods of its own that usually differ sharply from the more widely known methods of complex algebraic geometry. This feature is particularly apparent when studying the basic topological and geometric properties of real algebraic sets; the rich algebraic structures are usually hidden and cannot be recovered from the topology. The use of arc spaces and additive invariants partially obviates this disadvantage. Moreover, these methods are often parallel to the basic approaches of complex algebraic geometry.

Our presentation contains the construction of local topological invariants of real algebraic sets by means of algebraically constructible functions. This technique is extended to the wider family of arc-symmetric semialgebraic sets. Moreover, the latter family defines a natural topology that fills a gap between the Zariski topology and the euclidean topology.

In real equisingularity theory, Kuo's blow-analytic equivalence of real analytic function germs provides an equivalence relation that corresponds to topological equivalence in the complex analytic set-up. Among other applications, arc-symmetric geometry, via the motivic integration approach, gives new invariants of this equivalence, allowing some initial classification results.

The volume contains two courses and two survey articles that are designed for a wide audience, in particular students and young researchers.

***Résumé* (Espaces des arcs et invariants additifs en géométrie algébrique et analytique réelle)**

Nous présentons dans ce volume de nouvelles orientations en géométrie algébrique réelle qui reposent sur l'étude des espaces d'arcs et des invariants additifs d'ensembles algébriques réels. En général, la géométrie algébrique réelle utilise des méthodes qui lui sont propres et qui diffèrent d'habitude beaucoup des méthodes plus largement connues de la géométrie algébrique complexe. Ce trait est particulièrement apparent dans l'étude des propriétés topologiques et géométriques de base des ensembles algébriques réels ; les structures algébriques fécondes sont d'habitude cachées et ne peuvent pas être retrouvées à partir de la topologie. L'utilisation des espaces d'arcs et des invariants additifs remédie en partie à ce désavantage. De plus, ces méthodes sont souvent parallèles à des approches de base en géométrie algébrique complexe.

Notre présentation contient la construction d'invariants topologiques locaux des ensembles algébriques réels au moyen de fonctions algébriquement constructibles. Cette technique est étendue à la classe plus grande des ensembles symétriques par arc. De plus, cette dernière classe définit une topologie naturelle qui est intermédiaire entre la topologie de Zariski et la topologie euclidienne.

En théorie de l'équisingularité réelle, l'équivalence analytique après éclatement (blow-analytic equivalence) de Kuo fournit une relation d'équivalence pour les germes de fonctions analytiques réelles qui correspond à l'équivalence topologique dans le cadre analytique complexe. Parmi d'autres applications, la géométrie des ensembles symétriques par arc fournit, via l'approche par l'intégration motivique, de nouveaux invariants pour cette équivalence, et permet d'obtenir des premiers résultats de classification.

Ce volume contient deux cours et deux articles de synthèse qui sont conçus pour un large public, en particulier pour les étudiants et les jeunes chercheurs.

CONTENTS

RÉSUMÉS DES ARTICLES

Ces notes ont pour ambition d'expliquer les outils nécessaires à l'étude de la topologie des ensembles algébriques réels singuliers au moyen des fonctions algébriquement constructibles. La première section passe en revue les faits de base de la géométrie semialgébrique, notamment le théorème de triangulation et les résultats de trivialisation, cruciaux pour la notion d'entrelacs qui joue un rôle important dans ces notes. La deuxième section présente quelques résultats sur les ensembles algébriques réels, dont le théorème de Sullivan qui dit que la caractéristique d'Euler de l'entrelacs est paire, et l'existence d'une classe fondamentale. La troisième section est consacrée aux fonctions constructibles et algébriquement constructibles; l'outil principal qui rend ces fonctions utiles est l'intégration par rapport à la caractéristique d'Euler. On donne une idée de la façon dont les fonctions algébriquement constructibles donnent des invariants topologiques combinatoires qui permettent de caractériser les ensembles algébriques réels de petite dimension.

Les ensembles symétriques par arcs d'une variété analytique réelle sont les sous-ensembles vérifiant la condition : un arc analytique rencontre l'ensemble uniquement en des points isolés, ou bien est entièrement contenu dans l'ensemble. Les ensembles semi-algébriques symétriques par arcs d'un espace affine forment une famille contenant toutes les composantes connexes (même celles analytiques) des ensembles algébriques réels. En prenant cette famille de parties pour collection de fermés, on obtient une topologie noethérienne $\mathcal{AR}$ sur $\mathbb{R}^n$ plus fine que la topologie de Zariski. On montre que la topologie AR possède des propriétés similaires à celle de Zariski dans le cas algébrique complexe. On en déduit de nouvelles méthodes topologiques en géométrie algébrique réelle. Comme application nous montrons notamment que toute application injective et régulière d'un ensemble algébrique réel dans lui-même est surjective. C'est une version réelle du théorème d'Ax du cas algébriquement clos. Nous donnons aussi une preuve d'un résultat de Kucharz : toute classe d'homologie, à coefficients dans $\mathbb{Z}_2$, d'une variété de Nash compacte, se réalise comme classe fondamentale d'un sous-ensemble semi-algébrique symétrique par arcs.

Les fonctions algébriquement constructibles établissent un lien entre l'algèbre réelle et la topologie des ensembles algébriques réels. Dans cet article on présente l'historique, les définitions, les propriétés basiques et des caractérisations algébriques des fonctions algébriquement constructibles, ainsi qu'une description de l'obstruction locale pour qu'un espace topologique soit homéomorphe à un ensemble algébrique réel.

Nous étudions les fonctions et applications qui deviennent analytiques après composition avec un nombre localement fini d'éclatements. Le but principal de cet article est de donner de cette théorie un panorama indépendant contenant l'historique, des motivations, ainsi que des résultats récents et des questions ouvertes.

ABSTRACTS

The aim of these notes is to present the material needed for the study of the topology of singular real algebraic sets via algebraically constructible functions. The first chapter reviews basic results of semialgebraic geometry, notably the triangulation theorem and triviality results which are crucial for the notion of link, which plays an important role in these notes. The second chapter presents some results on real algebraic sets, including Sullivan's theorem stating that the Euler characteristic of a link is even, and the existence of a fundamental class. The third chapter is devoted to constructible and algebraically constructible functions; the main tool which makes these functions useful is integration against Euler characteristic. We give an idea of how algebraically constructible functions give rise to combinatorial topological invariants which can be used to characterize real algebraic sets in low dimensions.

Arc-symmetric subsets of a real analytic manifold are the subsets which satisfy the following test: given an analytic arc, then either the arc meets the set at isolated points or it is entirely included in the set. Arc-symmetric semialgebraic subsets of an affine space form a family which contains all connected (even analytic) components of real algebraic sets. Taking the sets of this family as closed sets we obtain a noetherian topology $\mathcal{AR}$ on $\mathbb{R}^n$, stronger than the Zariski topology. We show that the $\mathcal{AR}$ topology has similar properties to the Zariski one in the complex algebraic case, consequently we obtain some new topological methods in the real algebraic geometry. As an application we prove that injective regular self-maps of real algebraic sets are surjective, this is a real version of an analogous theorem of Ax for algebraically closed fields. We

give a proof of a result of Kucharz that the homology classes with $\mathbb{Z}_2$ coefficients of compact Nash manifolds can be realised by the fundamental classes of arc-symmetric semialgebraic sets.

Algebraically constructible functions connect real algebra with the topology of algebraic sets. In this survey we present some history, definitions, properties, and algebraic characterizations of algebraically constructible functions, and a description of local obstructions for a topological space to be homeomorphic to a real algebraic set.

We study function and map germs which become real analytic after composing with a locally finite number of blowing-ups. The main purpose of this article is to give a reasonably self-contained survey on this topic, including historical details concerning the development of this area, motivation, recent results, and important open problems.

INTRODUCTION

The space of analytic arcs on an algebraic variety was already considered in the works of O. Zariski and J. Nash in the context of resolution of singularities. In the 90's, M. Kontsevich introduced the technique of motivic integration on the space of arcs, which gave remarkable results on Hodge numbers and the McKay correspondence. Kontsevich's method was then developed by J. Denef and F. Loeser who introduced new invariants of algebraic varieties. Denef and Loeser applied the method of motivic integration to singularity theory, constructed a "motivic" Milnor fiber, and gave new interpretations of many classical invariants in terms of arc-spaces.

Independently, in real algebraic and analytic geometry, analytic arcs appeared in the singularity theory of real analytic functions and in the study of topological properties of real algebraic and semialgebraic sets. Recent years brought further progress in this area sparked by an application of the ideas of motivic integration. The main purpose of the meeting in Aussois was to present this development to young researchers. Therefore, the meeting consisted of introductory courses and of more advanced talks on recent results.

This volume contains the notes of two courses of this meeting:

- ▹ Real algebraic sets, [**C**]. This is an introduction to the real algebraic and semialgebraic sets that focusses on their topological properties.
- ▹ Arc-symmetric sets and arc-analytic mappings, [**KP**]. This is an introduction to arc-symmetric semialgebraic sets with a self-contained detailed account of recent results.

And two survey papers:

- ▹ on recent results on the topology of real algebraic sets and their relation to algebra, [**MP**].
- ▹ on blow-analytic equivalence in the sense of Kuo. This paper presents in particular the use of arc-spaces as a new technique in this area, [**FP**].

In this introduction we explain the relations among the four parts of the volume. We present some historical comments, motivation and the updated state of the art of the domain. We also wanted to indicate possible developments and open problems.

The introduction is divided according to the notions and concepts which appear in the different papers of the volume.

0.1. Arc-symmetric semialgebraic versus real algebraic sets

Arc-symmetric semialgebraic sets were introduced at the end of the 80's by K. Kurdyka, [**19**]. By definition, the arc-symmetric subsets of a real analytic manifold are the sets satisfying: given an analytic (parametrized) arc, then either the arc meets the set at isolated points or it is entirely included in the set. The arc-symmetric semialgebraic subsets define a topology that fills a gap between the Zariski topology and the strong topology. The strong topology is particularly poor in the real algebraic context, in contrast to complex algebraic geometry where it plays a much more meaningful role: irreducible sets are connected in the strong topology, the family of complex algebraic sets is stable by taking images of proper maps, etc. .

Arc-symmetric semialgebraic sets, though defined by local properties, share most of the global geometric properties with algebraic sets and may be used in many geometric local-global arguments. The natural "analytically rigid" components of a real algebraic set are arc-symmetric and semialgebraic, for instance the connected components or, more generally, the (locally) analytic components. However arc-symmetric semialgebraic sets are not necessarily locally analytic (see [**KP**]), and hence the arc-symmetric components form a finer class than the analytic components. Irreducible arc-symmetric semialgebraic sets, see [**19**] or section 1 of [**KP**], are in a sense arc-connected by analytic arcs, and turn out to be closely related to the Nash sheets considered in [**24**].

Since arc-symmetric semialgebraic sets are "less rigid" than algebraic ones, they are often better adapted for various geometric arguments. For instance, by Thom's representability theorem, each mod 2 homology class of a nonsingular compact real algebraic variety has an arc-symmetric representative, see [**17**] or section 6 of [**KP**]. On the other hand, the question of representability of mod 2 homology classes by algebraic sets is by far more difficult, see section 11.3 of [**6**].

In late 60's J. Ax [**1**], using model theory, proved that any injective regular self-map of a variety over an algebraically closed field is surjective. Then A. Borel [**9**] gave a topological proof of this result for complex algebraic and non-singular real algebraic varieties.

The use of arc-symmetric semialgebraic sets allows one to show, [**20**], an analogous result for real possibly singular varieties, generalizing, in this set-up, the theorems of Ax and Borel. We present this application in [**KP**], section 5.

0.1.1. Topological properties of arc-symmetric sets. — In the beginning of the 70's Sullivan noticed that the link of every point in a real algebraic variety has even (topological) Euler characteristic. After this discovery the Euler characteristic played an important role in the study of the local topology of real algebraic varieties. Benedetti-Dedo and Akbulut-King showed that a 2-dimensional compact triangulable space is homeomorphic to a real algebraic set if and only if it is mod 2 Euler (*i.e.* the link of every point has even Euler characteristic). For sets of dimension 3 there are four more such local topological conditions that characterize real algebraic sets, as shown by S. Akbulut and H. King in [**2**]. Moreover these conditions can be expressed in terms of the Euler characteristic of links, see [**10**].

Recent years brought a more formal approach, based on integration with respect to the Euler characteristic, the algebraically constructible functions, [**21**]. *An algebraically constructible function* on a real algebraic variety is, by definition, the Euler characteristic of the fibers of a regular map. Algebraically constructible functions are stable by many natural operations inherited from sheaf theory: sum, product, pullback, pushforward and duality. Duality can be expressed by a topological *link operator*, the constructible function counterpart of the topological construction of a link, and half of the link operator preserves the algebraically constructible functions, generalizing Sullivan's result. The algebraically constructible function formalism led to the construction of many more local topological invariants of real algebraic sets, for instance $2^{43} - 43$ independent invariants just for the sets of dimension 4, cf. [**22**]. This formalism turned out to be closely related to real algebra, more precisely, to the theory of real quadratic forms. By [**26**] and [**11**] the algebraically constructible functions on a real algebraic variety coincide with sums of signs of polynomials. In this volume an introduction to algebraically constructible functions is presented in [**C**] and further developments in [**MP**].

It was already noticed in [**21**] and then developed by I. Bonnard, [**8**], that the formalism of algebraically constructible functions has an "arc-symmetric" counter-part, *Nash constructible functions*. This shows that arc-symmetric semialgebraic sets are Euler mod 2 and satisfy Akbulut and King's conditions and there are more similarities between arc-symmetric semialgebraic sets and algebraic sets. This is discussed in [**KP**], section 4.

On the other hand the relation to the real algebra is more complex and technical. Nash constructible functions are sums of signs of blow-Nash semialgebraic functions, *i.e.* functions that become Nash (analytic with semi-algebraic graph) after blowings-up with smooth centers. The interested reader is invited to study [**8**].

0.1.2. Additive invariants. — An additive invariant, $X \to e(X)$, defined on algebraic varieties over a field k, takes values in an abelian group, depends only on the isomorphism class, and satisfies the additivity relation

$$e(X) = e(X \setminus Y) - e(Y)$$

for Y a closed subvariety of X. *A generalized Euler characteristic* is an additive invariant that satisfies

$$e(X \times Y) = e(X)e(Y).$$

For complex algebraic varieties the Hodge-Deligne polynomial is a nontrivial example of a generalized Euler characteristic.

The universal Euler characteristic is a purely formal construction that takes a variety over k to its class in the Grothendieck ring $K_0(Var_k)$. Kontsevich's motivic measure on the arc space of a complex algebraic variety takes values in the completion of the localized Grothendieck ring (cf. [**12**]). This completion is taken with respect to the virtual dimension, which makes sense if the varieties which are equivalent in the

Grothendieck ring have the same dimension. This is of course the case for complex algebraic varieties, since the degree of the Hodge-Deligne polynomial of a variety equals its dimension.

A similar formal construction can be carried out for real algebraic varieties. Then the following questions arise: Are there many generalized Euler characteristics? Do they determine the dimension? (If yes, then again the completion with respect to the virtual dimension in the corresponding Grothendieck ring of real algebraic varieties makes sense.) The simplest generalized Euler characteristic is the topological Euler characteristic with compact supports

$$\chi^c(X) := \sum_k (-1)^k \dim_{\mathbb{Z}_2} H^k_c(X;\mathbb{Z}_2).$$

(For locally compact constructible sets in complex algebraic geometry $\chi^c(X) = \chi(X)$ so one can drop the support requirement. This is due to duality and does not hold for arbitrary semialgebraic sets). χ^c is a topological invariant and it is in a sense the only one. Any other homeomorphism invariant generalized Euler characteristic defined on semialgebraic sets factors through it, [**29**]. On the other hand, evidently, $\chi^c(X)$ does not determine the dimension of X.

A new generalized Euler characteristic, the virtual Poincaré polynomial

$$\beta(X,t) = \sum \beta_i(X) t^i$$

was constructed in [**22**]. Its existence was independently announced in [**31**]. The coefficients $\beta_i(X)$ are called "the virtual Betti numbers" since

$$\beta_i(X) = \dim_{\mathbb{Z}_2} H^i(X;\mathbb{Z}_2)$$

if X is compact and non-singular. The existence of the virtual Poincaré polynomial follows from Bittner's presentation [**5**] of the Grothendieck ring with classes of projective non-singular varieties as generators and relations coming from blowings-up with smooth centers. Bittner's result uses the weak factorization theorem for birational mappings. The existence of virtual Betti numbers can be also showed by the method of Guillen and Navarro-Aznar [**13**].

As showed by G. Fichou [**14**] the virtual Betti numbers and the virtual Poincaré polynomial can be defined for set-theoretic combinations of compact arc-symmetric semialgebraic sets. (We discuss this class of semialgebraic sets in sections 3 and 4 of [**KP**].) It is clear that the virtual Betti numbers of real algebraic sets are isomorphism invariants and that they are not homeomorphism invariants. As Fichou shows the virtual Betti numbers are Nash isomorphism (*i.e.* analytic isomorphism with semialgebraic graph) invariants. Fichou's work is crucial for the application of arc spaces to blow-analytic equivalence.

0.2. Blow-analytic and arc-analytic mappings

A function $f : X \to \mathbb{R}$ defined on a real analytic space X is called *arc-analytic*, cf. [**19**], if for every real analytic $\gamma : (-1,1) \to X$ the composition $f \circ \gamma$ is analytic.

An arc-analytic function with subanalytic graph is continuous, cf. [**3**]. Many classical examples in calculus, as

$$f(x,y) = \frac{x^3}{x^2+y^2}, \quad f(0) = 0,$$

are arc-analytic but not analytic. This is in contrast to the complex case, where a meromorphic locally bounded function on a normal complex analytic space has to be complex analytic.

A semialgebraic function $f : X \to \mathbb{R}$, defined on a real algebraic set X is called *blow-Nash* if it can be made Nash after composing with a finite sequence of blowings-up with smooth nowhere dense centers. It was conjectured by Kurdyka and shown in [**3**] by Bierstone and Milman, that f is blow-Nash if and only if f is arc-analytic and the graph of f is semialgebraic.

Similarly one defines arc-analytic mappings between two arc-symmetric sets. Arc-analytic semialgebraic mappings form a good class of morphisms between arc-symmetric semialgebraic sets, cf. [**19**] and [**KP**]. Blow-analytic mappings are used in the study of blow-analytic equivalence in the sense of Kuo, cf. [**18**] and [**FP**].

0.2.1. Blow-analytic equivalence of real-analytic function germs. — The classification of real analytic function germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ requires a good natural equivalence relation. When one needs a "crude" equivalence that does not have continuous moduli (*i.e.* analytic families of germs that are all non-equivalent), real singularists face a real problem. For complex analytic function germs with isolated singularities topological equivalence (that does not have continuous moduli) has many interesting properties and such invariants as the Milnor number or the monodromy. But topological equivalence is by far too poor for real analytic function germs: for two variables germs it is characterized by the number of branches of the zero set at the origin plus the distribution of signs on the complement. Other natural equivalences, as C^1 or bi-lipschitz, have continuous moduli.

Blow-analytic equivalence was proposed by T.-C. Kuo at the beginning of the 80's. Two real analytic function germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are called *blow-analytically equivalent* if there exists a homeomorphism germ $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, such that $f = g \circ \varphi$ and φ becomes real analytic after suitable modifications of both the source and the target (this means intuitively that f and g have analytically isomorphic resolutions). There are several versions of this notion that put slightly different conditions on φ. We refer to [**FP**] and the bibliography therein for a more technical discussion. Kuo [**18**] showed that blow-analytic equivalence does not admit continuous moduli for families of isolated singularities (this is false for complex singularities, real analytic isomorphisms are less "rigid").

Families of real isolated singularities admitting natural equiresolutions, *e.g.* toric, are blow-analytically trivial. It is less clear, however, how to decide whether two real analytic function germs are not blow-analytically equivalent. In recent years the first such invariants of blow-analytic equivalence were constructed. They use arc spaces.

0.2.2. Arc spaces and blow-analytic equivalence. — For a real analytic function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and a positive integer k we denote by $\mathcal{X}_k(f)$ the set