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RIEMANN'S *COMMENTATIO MATHEMATICA*, A REASSESSMENT

Alberto Cogliati

Abstract. — Starting with the publication of Riemann's *Gesammelte Mathematische Werke* in 1876, the *Commentatio Mathematica* has attracted considerable interest among mathematicians and historians. Nonetheless, there appears to be no consensus on the most appropriate approach to the interpretation of the paper, namely on its relationship with Riemann's *Habilitationsvortrag*.

This article represents a contribution to such an interesting debate. Special attention is paid to the *pars secunda* of Riemann's *Commentatio*. In particular, the focus is on the interpretation of a certain trinomial expression from which Riemann claimed that an understanding of his "curvature tensor" could be achieved.

Résumé (Un réexamen du *Commentatio Mathematica* de Riemann)

À commencer avec la publication des œuvres de Riemann en 1876, la *Commentatio Mathematica* de Riemann a suscité beaucoup d'intérêt parmi les mathématiciens et les historiens. Il semble pourtant qu'aucun consensus sur la bonne lecture de ce texte en rapport avec le *Habilitationsvorstrag* de Riemann ne se soit dégagé. Cet article contribue à ce débat intéressant. Nous prêtons particulièrement attention à la *pars secunda*, nous centrons notre interprétation autour d'une certaine expression trinomiale dont Riemann prétend qu'elle permet de comprendre le « tenseur de courbure ».

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1. INTRODUCTION

According to an established historiographical notion, Riemann's *Commentatio Mathematica* submitted in 1861 for a prize to the *Académie des Sciences* of Paris can be regarded as a further development and as a more explicit elaboration of the revolutionary ideas which Riemann himself set forth in 1854 on the occasion of his habilitation lecture [Riemann 1854] on the hypotheses laying at the basis of geometry.

Such an interpretation attitude was first propounded in 1876 by H. Weber, the editor with R. Dedekind of Riemann's *Werke*. At the beginning of his highly detailed commentary notes appended to Riemann's paper, he wrote that "These investigations are most intimately connected to the paper Ueber die Hypothesen welche der Geometrie zu Grunde liegen".[1] Accordingly, Weber devoted a large amount of his attempt at clarifying Riemann's ideas to a discussion of their geometrical content, by putting special emphasis upon the introduction of a special type of coordinates (later known as normal Riemannian coordinates) and of second order differentials which represented an indispensable technical means for the geometrical interpretation of the four-index quantities introduced by Riemann in his discussion of the equivalence problem of two quadratic differential forms.

In 1869, before Riemann's *Commentatio* was published for the first time, and thus independently of it, Elwin Bruno Christoffel and Rudolf Lipschitz who in reason are considered the founding fathers of what later became known as tensor calculus, tackled the very same problem with which Riemann had confronted himself in the *Commentatio*, providing a detailed analysis of the equivalence problem. They both obtained necessary and sufficient conditions guaranteeing the existence of a coordinate transformation which carries one quadratic differential form into another.

Their contributions [Christoffel 1869] and [Lipschitz 1869], both appeared in the same volume of *Crelle's Journal*, were characterized by distinct

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1 *Diese Untersuchungen hängen aufs Innigste zusammen mit der Abhandlung Ueber die Hypothesen welche der Geometrie zu Grunde liegen* [Riemann 1876, p. 384]. It is interesting to note that in the second edition of [Riemann 1876] Weber rephrased this statement as follows: "These investigations contain the analytical details of the results outlined in the paper Ueber die Hypothesen welche der Geometrie zu Grunde liegen." *Diese Untersuchungen enthalten die analytischen Ausführungen zu den in der Abhandlung "Ueber die Hypothesen welche der Geometrie zu Grunde liegen" angedeuteten Resultaten.* See [Riemann 1876, 2nd edition, p. 405].

approaches and quite dissimilar emphases upon the significance to be attributed to the investigations contained therein. While Christoffel's analysis can aptly be regarded as a chapter in the theory of algebraic invariants, Lipschitz's contribution revealed a closer link with Gauß' theory of curved surfaces and with Riemann's n-dimensional metric geometry.

Lipschitz was particularly explicit in singling out the main motivation at the basis of his research by providing, from the very start, a geometrical interpretation of a quadratic differential form as the square of the line element of a given (in general, n-dimensional) surface. He wrote: "Correspondingly, the analytic expression for the square of the line-element of a given surface in two independent variables, upon which Gauss has based his disquisitiones generales circa superficies curvas, is a quadratic form of two differentials. Gauss's researches on this subject have found an outstanding application in Riemann's posthumously published essay über die Hypothesen, welche der Geometrie zu Grunde liegen. Owing to this, the interest in the forms, and especially in those of second degree and in any number of differentials, has considerably increased. A key point in Riemann's investigations is represented by the conveyence of criteria guaranteing the existence of a change of variables which transforms one such form into another one which is sum of the square of the differentials of the new variables. The present work aims at providing a direct answer to this question in the case of second degree forms of n differentials whose determinant does not vanish identically under the imposed conditions. In this sense, it represents an extension of the theory of Gaussian curvature".[2]

After the publication of [Riemann 1876], in [Lipschitz 1876] Lipschitz came back to the topic which he had tackled in 1869 by interpreting those

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2 *Demgemäss ist der analytische Ausdruck für das Quadrat des Linearelements einer gegebenen Fläche in zwei unabhängigen Veränderlichen, den Gauss den* disquisitiones generales circa superficies curvas *zu Grunde gelegt hat, eine quadratische Form von zwei Differentialen. Die hierauf bezüglichen Gaussischen Forschungen haben in dem aus Riemann's Nachlasse publicirten Aufsatz über die Hypothesen, welche der Geometrie zu Grunde liegen, eine merkwürdige Anwendung gefunden, und es ist damit das Interesse an der bezeichneten Gattung von Formen, und in's besondere an denen des zweiten Grades und von beliebig vielen Differentialen, bedeutend gewachsen. Einen Angelpunkt von Riemann's Untersuchungen bildet die Ermittelung der Criterien, von denen es abhängt, ob eine derartige Form durch Einführung eines neuen Systems von Variabeln in eine Form transformirt werden könne, welche das Aggregat der Quadrate von den Differentialen der neuen Variabeln ist.* [...] *Die gegenwärtige Arbeit verfolgt vornehmlich den Zweck, die gestellte Frage in Betreff der Formen zweiten Grades von n Differentialen, deren Determinante zufolge der getroffenen Einschränkung nicht identisch verschwindet, direct zu beantworten, und in diesem Sinne die Theorie des Gaussischen Krümmungsmasses auszudehnen* [Lipschitz 1869, p. 71 and p. 73].