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SEMI-CLASSICAL LIMIT OF THE LOWEST EIGENVALUE OF A SCHRÖDINGER OPERATOR ON A WIENER SPACE: I. UNBOUNDED ONE PARTICLE HAMILTONIANS

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — We study a semi-classical limit of the lowest eigenvalue of a Schrödinger operator on a Wiener space. The Schrödinger operator is a perturbation of the second quantization operator of an unbounded self-adjoint operator by a C^3-potential function. This result is an extension of [**1**].

***Résumé* (Limite semi-classique de la plus petite valeur propre d'un opérateur de Schrödinger sur l'espace de Wiener: cas d'un Hamiltonien non borné à une particule.)**
Nous étudions le comportement semi-classique de la plus petite valeur propre d'un opérateur de Schrödinger sur l'espace de Wiener. L'opérateur de Schrödinger est obtenu par perturbation de l'opérateur de seconde quantification associé à un opérateur non-borné autoadjoint donné par un potentiel C^3. Ce résultat est une extension de [**1**].

1. Introduction

In [**1**], we studied the semi-classical limit of the lowest eigenvalue of Schrödinger operators which are perturbations of the number operator. In that case, one particle Hamiltonian (the coefficient operator of the second order differential operator) is identity operator. However, we need to study the case where the coefficient operator is unbounded to study $P(\phi)$-type Hamiltonians. For example, the typical coefficient operator is $\sqrt{m^2 - \Delta}$, where $m > 0$ and Δ is the Laplace-Bertlami operator on $\mathbb{R}$. In this paper, we study the asymptotics of the lowest eigenvalue of a Schrödinger operator in the case where the coefficient operator is unbounded linear operator and the potential function is C^3. In $P(\phi)$-type model cases, the potential functions are defined by using a renormalization and they are not continuous. In [**2**], we studied

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Schrödinger operators on path spaces over Riemannian manifolds. In that case, the differential operators are variable coefficient ones and the coefficient operators are not bounded linear because they contain stochastic integrals. Moreover, the dependence on the path of the coefficients are discontinuous in the natural topology. The discontinuity comes from the discontinuity of solutions of stochastic differential equations as a functional of Brownian motion. Thus, we need to consider two kind of discontinuity for potential functions and coefficient operators in that case. But, the difficulties are different from that of the $P(\phi)$-type potentials. We will study semi-classical limit of the lowest eigenvalue of a $P(\phi)_2$-Hamiltonian on a finite interval in [**3**].

2. Preliminaries

Let (W, H, μ) be an abstract Wiener space. That is,

(i) H is a separable Hilbert space and W is a separable Banach space. Moreover H is continuously and densely embedded into W,

(ii) μ is the unique Gaussian measure on W such that for any $\varphi \in W^*$,

$$\int_W e^{\sqrt{-1}\varphi(w)} d\mu(w) = e^{-\frac{1}{2}\|\varphi\|_H^2}.$$

Here we use the natural inclusion and the identification by the Riesz theorem $W^* \subset H^* \simeq H$.

In this paper, we assume that W is a Hilbert space. This is equivalent to that there exists a positive self-adjoint trace class operator S such that W is a completion of H with respect to the Hilbert norm $\|\sqrt{S}h\|_H$. That is, $\|h\|_W = \|\sqrt{S}h\|_H$ for all $h \in H$. We denote the sets of bounded linear operators, Hilbert-Schmidt operators, trace class operators on H by $L(H), L_1(H), L_2(H)$. Also we denote their operator norms, trace norms, Hilbert-Schmidt norms by $\| \|, \| \|_1, \| \|_2$, respectively. For $\lambda > 0$, we define the new measure μ_λ on W by $\mu_\lambda(E) = \mu\left(\sqrt{\lambda}E\right)$ $(E \subset W)$. Now we define our Schrödinger operators.

Definition 2.1. — *Let A be a strictly positive self-adjoint operator on H. That is, we assume that* $\inf \sigma(A) > 0$, *where $\sigma(A)$ denotes the spectral set of A. We denote $c_A = \inf \sigma(A^2)$. We denote by $\mathfrak{F}C_A^\infty(W)$ the space of all smooth cylindrical functions $f(w) = F(\varphi_1(w), \ldots, \varphi_n(w))$ $(F \in C_b^\infty(\mathbb{R}^n), \varphi_i \in W^* \cap_{n\in\mathbb{N}} \mathrm{D}(A^n))$. For such a f, we define $Df(w) = \sum_{i=1}^n \partial_i F(w)\varphi_i \in H$. Here we use the identification $\varphi_i \in W^* \subset H^* \simeq H$ and $\partial_i F(w)$ denotes the partial derivative with respect to the i-th variable. Moreover we define $D_A f(w) = \sum_{i=1}^n \partial_i F(w) A\varphi_i$. We define a Dirichlet form on $L^2(W, d\mu_\lambda)$ by $\mathcal{E}_{\lambda,A}(f, f) = \int_W \|D_A f(w)\|_H^2 d\mu_\lambda(w)$. $-L_{\lambda,A}$ denotes the generator. Let V be a real-valued measurable function on W such that $V \in \cap_{\lambda>0} L^1(W, \mu_\lambda)$. Under the assumption that for all $\lambda > 0$, $\mathcal{E}_{\lambda,A,V}(f, f) =$*

$\mathscr{E}_{\lambda,A}(f,f)+\int_W \lambda^2 V(w)f(w)^2 d\mu_\lambda(w)$ $(f \in \mathfrak{F}C_A^\infty(W))$ *is a lower bounded symmetric form, we denote the generator of the smallest closed extension by* $-L_{\lambda,A,V}$. *Also let* $E_0(\lambda,A,V)=\inf\sigma(-L_{\lambda,A,V})$.

Remark 2.2. — (1) $-L_{\lambda,A}$ can be viewed as the second quantization of A^2 on H. Let $H=H^{1/2}(\mathbb{R})$ be the Hilbert space with the norm $\|h\|_H^2=\int_{\mathbb{R}}|(m^2-\Delta)^{1/4}h(x)|^2dx$, where $m>0$. Consider $A=(m^2-\Delta)^{1/4}$ on H. In this case, $-L_{1,A}$ is the time 0 field free Hamiltonian in $P(\phi)_2$-model. However note that $-L_{1,A}$ is usually identified with the second quantization of $\sqrt{m^2-\Delta}$ on $H^*=H^{-1/2}(\mathbb{R})$. See also Example 3.3.
(2) In [**1, 5**], the Schrödinger operator with semi-classical parameter λ is defined in a different way. Let $V_\lambda(w)=\lambda V\left(\frac{w}{\sqrt{\lambda}}\right)$. The semi-classical limit of $-L_{1,A}+V_\lambda$ on $L^2(W,d\mu)$ is studied in the above papers. However note that this operator is unitarily equivalent to $-L_{\lambda,A,V}/\lambda$ on $L^2(W,\mu_\lambda)$. We adopt the similar definition to $-L_{\lambda,A,V}$ in the case of Schrödinger operators on path spaces over Riemannian manifolds because the scaling $w/\sqrt{\lambda}$ can not defined on the curved spaces but the measure corresponding to μ_λ can be defined on curves spaces too. See Remark 5.3 in [**1**] and [**2**].

Let us introduce the following assumptions on potential functions of Schrödinger operators.

Assumption 2.3. — The following assumptions (A1), (A2) are standard in semi-classical analysis. (A4) assures that the symmetric form $\mathscr{E}_{\lambda,A,V}$ is bounded from below by Corollary 2.8 (2). Note that (A5) implies that A is an unbounded operator.

(A1) V is a C^2-function on H. Let $U(h)=\frac{1}{4}\|Ah\|_H^2+V(h)$ $(h\in \mathrm{D}(A))$. Then $\min_{h\in\mathrm{D}(A)}U(h)=0$ and the zero point set is a finite set $N=\{h_1,\ldots,h_n\}$.
(A2) $\frac{1}{2}D^2U(h_i)=\frac{1}{4}A^2+K_i$ is a strictly positive self-adjoint operator on H, where $K_i=\frac{1}{2}D^2V(h_i)\in L(H,H)$.
(A3) V can be extended to a C^3-function on W such that for any $R>0$ and $0\le k\le 3$

$$\sup\left\{\|D^kV(w)\|_{L(W\times\cdots\times W,\mathbb{R})}\mid \|w\|_W\le R\right\}\le C(R)<\infty.$$

(A4) V can be extended to a continuous function on W and there exists $p>1$ such that

$$\limsup_{\lambda\to\infty}\lambda^{-1}\log\int_W e^{-\frac{2p\lambda}{c_A}V(w)}d\mu_\lambda(w)<\infty,$$

(A5) There exists $\gamma_0>1$ such that $A^{-\gamma_0}\in L_2(H)$.

For $r>0$ and $z\in W, k\in H$, we denote $B_r(z)=\{w\in W\mid \|w-z\|_W\le r\}$ and $B_{r,H}(k)=\{h\in H\mid \|h-k\|_H\le r\}$.

Lemma 2.4. — (1) *Suppose that* (A4) *holds or* $\inf\{V(h)\mid h\in H\}>-\infty$. *Then we have* $\lim_{\|h\|_H\to\infty}\left(\frac{c_A}{4}\|h\|_H^2+V(h)\right)=+\infty.$

(2) *Assume* (A1)*, the same assumptions in* (1) *and for any* $L > 0$, $\sup\{|V(h)| \mid \|h\|_H \leq L\} < \infty$. *Then for any* $\varepsilon > 0$,

$$\kappa(\varepsilon) := \inf\left\{U(h) \mid h \in \{\cup_{i=1}^n B_\varepsilon(h_i)\}^c\right\} > 0.$$

Proof. — (1) If $\inf\{V(h) \mid h \in H\} > -\infty$, the statement is trivial. We assume (A4). Let C be a positive number such that $\limsup_{\lambda\to\infty} \lambda^{-1} \log \int_W e^{-\frac{2p\lambda}{c_A}V} d\mu_\lambda < C$. Take $R > 0$. Then for sufficiently large λ, we have

$$\begin{aligned}
&\frac{1}{\lambda}\log\int_W \exp\left(-\frac{2p\lambda}{c_A}\left(R\wedge V(w)\vee(-R)\right)\right) d\mu_\lambda(w) \\
&\quad\leq \frac{1}{\lambda}\log\left(\int_W \left(e^{-\frac{2p\lambda}{c_A}R} + \exp\left(-\frac{2p\lambda}{c_A}(V(w)\vee(-R))\right) d\mu_\lambda(w)\right)\right) \\
&\quad\leq \frac{1}{\lambda}\log\left(e^{\lambda C} + e^{-\frac{2p\lambda}{c_A}R}\right) \leq C + \frac{\log 2}{\lambda}.
\end{aligned}$$

By the Large deviation estimate, we have

$$\sup_h\left(-\frac{1}{2}\|h\|_H^2 - \frac{2p}{c_A}\left((-R)\vee V(h)\wedge R\right)\right) \leq C.$$

Since R is an arbitrary number, we get

$$-\frac{c_A}{4}\|h\|_H^2 - pV(h) \leq \frac{C\cdot c_A}{2} \quad \text{for all } h\in H.$$

Suppose that there exists $\{h_n\}$ such that $\|h_n\|_H \to \infty$ and $\sup_n\left(\frac{c_A}{4}\|h_n\|_H^2 + V(h_n)\right) =: l < +\infty$. Then $\lim_{n\to\infty} V(h_n) = -\infty$. Hence

$$\frac{c_A}{4}\|h_n\|_H^2 + pV(h_n) = \frac{c_A}{4}\|h_n\|_H^2 + V(h_n) + (p-1)V(h_n) \leq l + (p-1)V(h_n) \to -\infty.$$

This is a contradiction. So we are done.
(2) By the result in (1), we need to prove that for sufficiently large positive number L,

$$\inf\{U(h) \mid h \in B_{L,H}(0) \cap (\cup_{i=1}^n B_\varepsilon(h_i))^c\} > 0.$$

Suppose that there exists $\{\varphi_l\} \subset B_{L,H}(0) \cap (\cup_{i=1}^n B_\varepsilon(h_i))^c$ such that $\lim_{l\to\infty} U(\varphi_l) = 0$. By the assumption, there exists a subsequence $\{\varphi_{l(i)}\}$ which converges to a certain element $\varphi_\infty \in H$ weakly. Since $\frac{1}{4}\|A\varphi_{l(i)}\|_H^2 = U(\varphi_{l(i)}) - V(\varphi_{l(i)})$, $\sup_i \|A\varphi_{l(i)}\|_H < \infty$ holds. Hence again by choosing a subsequence $\{\varphi_{p(i)}\}$, $A\varphi_{p(i)}$ also converges to some ϕ_∞ weakly. By the Banach-Saks theorem, we see that $\varphi_\infty \in \mathrm{D}(A)$ and $A\varphi_\infty = \phi_\infty$. On the other hand, since the embedding $H \subset W$ is compact, $\lim_{i\to\infty}\|\varphi_{p(i)} - \varphi_\infty\|_W = 0$ which implies $\lim_{i\to\infty} V(\varphi_{p(i)}) = V(\varphi_\infty)$. Since $\|A\varphi_\infty\|_H^2 \leq \liminf_{i\to\infty}\|A\varphi_{p(i)}\|_H^2$, we obtain $U(\varphi_\infty) \leq \liminf_{i\to\infty} U(\varphi_{p(i)}) = 0$. This implies $\varphi_\infty \in N$ and $\varphi_{p(i)} \in B_\varepsilon(h_j)$ for some large i and $1 \leq j \leq n$. This is a contradiction. □

Lemma 2.5. — *Let A be a strictly positive self-adjoint operator and K be a trace class self-adjoint operator on H. Assume that A^2+K is also a strictly positive operator. Then $\sqrt{A^2+K}-A \in L_1(H)$ and*

$$\left\|\sqrt{A^2+K}-A\right\|_1 \leq \frac{\|K\|_1}{\min\left\{\inf\sigma(\sqrt{A^2+K}),\inf\sigma(A)\right\}}.$$

Proof. — We prove this in three steps: (i) $A=I+T$ and T is a trace class operator, (ii) A is a bounded linear operator, (iii) General cases.

(i) We denote $S_1=\sqrt{A^2+K}$ and $S_0=A$. Note that $S_1-S_0=\sqrt{A^2+K}-A$ is a trace class operator. We denote the all eigenvalues and corresponding complete orthonormal system of S_1-S_0 by $\{\alpha_n\}$ and $\{e_n\}$. Then

$$\begin{aligned}|(Ke_n,e_n)| &= |\left((S_1^2-S_0^2)e_n,e_n\right)| \\ &= \left|\left(\left(S_1(S_1-S_0)+(S_1-S_0)S_1-(S_1-S_0)^2\right)e_n,e_n\right)\right| \\ &= |\alpha_n\left((S_1+S_0)e_n,e_n\right)| \\ &\geq |\alpha_n|\inf\sigma(S_1+S_0).\end{aligned}$$

This implies that

$$\|\sqrt{A^2+K}-A\|_1=\sum_{n=1}^{\infty}|\alpha_n|\leq\frac{\|K\|_1}{\inf\sigma(\sqrt{A^2+K}+A)}.$$

(ii) Let $\{u_m\}$ be all eigenvectors of K which is a c.o.n.s. of H. Set $P_mh=\sum_{i=1}^m(h,u_i)u_i$ and $A_m=\sqrt{P_mA^2P_m+P_m^{\perp}}$. Then $A_m^2\to A^2$, $A_m\to A$ converge strongly. On the other hand, $A_m^2+K=P_m(A^2+K)P_m+P_m^{\perp}(I_H+P_m^{\perp}KP_m^{\perp})P_m^{\perp}$. Hence for sufficiently large m, we have

$$\min\left\{\inf\sigma(\sqrt{A_m^2+K}),\inf\sigma(A_m)\right)\geq\min\left(\inf\sigma(\sqrt{A^2+K}),1/2,\inf\sigma(A)\right).$$

Since A_m-I_H is a trace class operator, by (i),

$$\|\sqrt{A_m^2+K}-A_m\|_1\leq\frac{\|K\|_1}{\min\left(\inf\sigma(A^2+K),\inf\sigma(A),1/2\right)}.$$

By taking the limit $m\to\infty$, we see that $\sqrt{A^2+K}-A\in L_1(H)$. Therefore again by the same argument as in (i), we can prove (ii).

(iii) Let $\chi_n(x)$ be a function such that $\chi_n(x)=1$ for $x\leq n$ and $\chi_n(x)=0$ for $x>n$. Then $\chi_n(A)$ is a projection operator which commutes with A. Let $A_n=A\chi_n(A)+(1-\chi_n(A))$ and $K_n=\chi_n(A)K\chi_n(A)$. Then

$$\begin{aligned}\sqrt{A^2+K_n}-A &= \sqrt{A^2\chi_n(A)+\chi_n(A)K\chi_n(A)}-A\chi_n(A) \\ &= \sqrt{A_n^2+K_n}-A_n\in L(\mathrm{Im}(\chi_n(A)))\end{aligned}$$

By (ii), we have

$$(2.1)\quad \|\sqrt{A^2+K_n}-A\|_1 \;\leq\; \frac{\|K_n\|_1}{\inf\sigma\left(\sqrt{A^2\chi_n(A)+\chi_n(A)K\chi_n(A)}+A\chi_n(A)\right)}$$
$$\leq\; \frac{\|K_n\|_1}{\min\left(\inf\sigma(\sqrt{A^2+K}),\inf\sigma(A)\right)}.$$

For $l>n>m$,

$$\begin{aligned}\left(\sqrt{A_n^2+K_n}-A_n\right)-\left(\sqrt{A_m^2+K_m}-A_m\right) &= \sqrt{A^2+K_n}-\sqrt{A^2+K_m}\\ &= \sqrt{A_l^2+K_n}-\sqrt{A_l^2+K_m}.\end{aligned}$$

This and (ii) implies that $\sqrt{A_n^2+K_n}-A_n$ converges in the trace norm. It is not difficult to check that the strong limit is equal to $\sqrt{A^2+K}-A$. Therefore, (2.1) implies the conclusion. □

Proposition 2.6. — *Let A be a strictly positive self-adjoint operator. For a trace class self-adjoint operator K on H and $h\in \mathrm{D}(A^2)$, we set*

$$V_{K,h}(w)=\frac{1}{4}\|Ah\|_H^2-\frac{1}{2}(A^2h,w)+(K(w-h),w-h)\,.$$

We assume that A^2+4K is a strictly positive self-adjoint operator and AKA can be extended to a trace class operator. Then $\mathcal{E}_{\lambda,A,V_{K,h}}$ is a symmetric form bounded from below and $E_0(\lambda,A,V_{K,h})=\lambda e(A,K)$ holds, where

$$(2.2)\qquad e(A,K)=\frac{1}{2}\mathrm{tr}\left(\sqrt{A^4+4AKA}-A^2\right).$$

Moreover it is the lowest eigenvalue of $-L_{\lambda,A,V_{K,h}}$ and the corresponding normalized positive eigenfunction is

$$\begin{aligned}\Omega_{\lambda,A,V_{K,h}}(w) &= \det\left(I_H+T_K\right)^{1/4}\\ &\quad\times\exp\left\{-\frac{\lambda}{4}\left(\left(A^{-1}\{A^4+4AKA\}^{1/2}A^{-1}-I_H\right)(w-h),(w-h)\right)\right\}\\ &\quad\times\exp\left(\frac{\lambda}{2}(h,w)-\frac{\lambda}{4}\|h\|_H^2\right),\end{aligned}$$

where $T_K=A^{-1}(\sqrt{A^4+4AKA}-A^2)A^{-1}$.

Proof. — If A is bounded linear operator, the proof is a straightforward calculation. Suppose that A is unbounded. Let A_n and K_n be the operators which are defined in the proof of (iii) in Lemma 2.5. Then $AK_nA=A_nK_nA_n$. Thus $\left(A^{-1}\{A^4+4AK_nA\}^{1/2}A^{-1}-I_H\right)\in L_1(H)\cap_k \mathrm{D}(A^k)$. Therefore for sufficiently large n, $\Omega_{\lambda,A,V_{K_n,h}}\in L^2(\mu_\lambda)$ and the simple calculation shows that

$$-L_{\lambda,A,V_{K_n,h}}\Omega_{\lambda,A,V_{K_n,h}}=\lambda e(A,K_n)\Omega_{\lambda,A,V_{K_n,h}}.$$

Letting $n \to \infty$, we have

$$-L_{\lambda,A,V_{K,h}}\Omega_{\lambda,A,V_{K,h}} = \lambda e(A,K)\Omega_{\lambda,A,V_{K,h}}.$$

To prove that $\lambda e(A,K) = \inf \sigma\left(-L_{\lambda,A,V_{K,h}}\right)$, we note that for any $f \in \mathfrak{F}C_A^\infty(W)$, it holds that

$$\begin{aligned}\mathscr{E}_{\lambda,A,V_{K,h}}(f,f) = \int_W \|D_A(f\Omega^{-1}_{\lambda,A,V_{K,h}})\|_H^2 \Omega_{\lambda,A,V_{K,h}}(w)^2 d\mu_\lambda(w)&\\ + \lambda e(A,K)\|f\|^2_{L^2(\mu_\lambda)}.&\end{aligned}$$ □

We use the following estimate to prove a lower bound in Lemma 3.4. We refer the reader to [**7, 12, 14**] for this estimate.

***Theorem 2.7* (NGS estimate).** — *Let $\mathscr{E}(f,f)$ be a closed form on $L^2(X,m)$, where $(X,\mathscr{F},m)$ is a probability space. Assume that there exists $\alpha > 0$ such that for any $f \in \mathrm{D}(\mathscr{E})$,*

$$\int_X f(x)^2 \log\left(f(x)^2/\|f\|^2_{L^2(X,m)}\right) dm(x) \le \alpha\mathscr{E}(f,f).$$

Then for any bounded measurable function V, it holds that

$$\mathscr{E}(f,f) + \int_X V(x)f(x)^2 dm(x) \ge -\frac{1}{\alpha}\log\left(\int_X e^{-\alpha V(x)}dm(x)\right)\|f\|^2_{L^2(X,m)}. \tag{2.3}$$

The following follows from the above estimate and Gross's logarithmic Sobolev inequality [**7**]: For any $f \in \mathfrak{F}C_I^\infty(W)$,

$$\int_W f(w)^2 \log\left(f(w)^2/\|f\|^2_{L^2(\mu_\lambda)}\right) d\mu_\lambda(w) \quad \le \quad \frac{2}{\lambda}\int_W \|Df(w)\|_H^2 d\mu_\lambda(w).$$

Originally NGS(=Nelson, Glimm, Segal) estimate (2.3) was proved by the hypercontractivity of the corresponding semigroup. See [**14**]. Corollary 2.8 (2) is proved by Lemma 4.5 in [**2**] which follows from Gross's log-Sobolev inequalities and finite dimensional approximations.

Corollary 2.8. — (1) *It holds that*

$$E_0(\lambda,A,V) \ge -\frac{\lambda c_A}{2}\log\left(\int_W \exp\left(-\frac{2\lambda}{c_A}V\right)d\mu_\lambda(w)\right).$$

(2) *Suppose that there exists a Hilbert-Schmidt operator T such that $A = I + T$. Then*

$$\begin{aligned}&E_0(\lambda,A,V)\\ &\quad\ge -\frac{\lambda}{2}\log\left\{\int_W \exp\left(-2\lambda V(w) - \lambda : (Tw,w) :_{\mu_\lambda} -\frac{\lambda}{2}\|Tw\|_H^2\right)d\mu_\lambda(w)\right\}\\ &\quad\quad +\frac{\lambda}{2}\log\det{}_{(2)}(I_H + T) - \frac{\lambda}{2}\mathrm{tr}\left(T^2\right).\end{aligned} \tag{2.4}$$

In (2.4), $:(Tw,w):_{\mu_\lambda}$ is defined by the limit $\lim_{n\to\infty}\left\{(P_nTP_nw,w)-\frac{1}{\lambda}\mathrm{tr}P_nTP_n\right\}$, where P_n is a projection on to a finite dimensional subspace of H such that $P_n \uparrow I_H$. $\det_{(2)}$ denotes the Carleman-Fredholm determinant.

3. Results

***Theorem 3.1* (Bounded case).** — *We assume that A is a bounded linear operator and satisfies the assumptions* (A1), (A2), (A3), (A4). *Then we have*

$$\lim_{\lambda\to\infty}\frac{E_0(\lambda,A,V)}{\lambda}=\min_{1\le i\le n}e(A,K_i). \tag{3.1}$$

In the unbounded case, we can prove the following. The assumption is too strong to cover the $P(\phi)$-type Hamiltonian. We will relax the assumptions and discuss such a case in a separate paper.

***Theorem 3.2* (Unbounded case).** — *Assume* (A5). *Let $\gamma\ge 1+\gamma_0$ and $S=A^{-2\gamma}$. Then AK_iA is a trace class operator and* (2.2) *is well-defined. Furthermore, we assume that* (A1), (A2), (A3), (A4) *hold. Then the asymptotics* (3.1) *holds.*

Example 3.3. — Let $I=[-\frac{l}{2},\frac{l}{2}]$ $(l>0)$ be an interval of $\mathbb{R}$. Let $-\Delta$ be the Laplacian with periodic boundary condition on $X=L^2(I\to\mathbb{R},dx)$. Let $m>0$. For $\alpha\in\mathbb{R}$, let $H^\alpha=D((m^2-\Delta)^{\alpha/2})$ and $\|h\|_{H^\alpha}=\|(m^2-\Delta)^{\alpha/2}h\|_X$.
(1) Let $H=H^{1/2}$. Then for any $\varepsilon>0$, we can take $W=H^{-\varepsilon}$. Let $0<\varepsilon<1/2$. Then using the inclusion and the identification $H^{1/2}\subset H^{\varepsilon}=(H^{-\varepsilon})^*$, we can see that μ satisfies that $\int_W {}_{H^{-\varepsilon}}(w,h)_{H^{\varepsilon}}^2\,d\mu(w)=\|(m^2-\Delta)^{-1/4}h\|_X^2$ for $h\in H$. Let $U:X\to H^{1/2}$ be the natural isometry operator and define $A=U(m^2-\Delta)^{1/4}U^{-1}$. This is a standard example in $P(\phi)_2$-model on finite interval. Let $P(u)=\sum_{k=0}^{2M}a_ku^k$ be a polynomial with real coefficients with $a_{2M}>0$. For $h\in H$, $\tilde{V}(h)=\int_I P(h(x))dx$ is well-defined by the Sobolev embedding theorem. However $H^{-\varepsilon}$ is the space of distribution and $P(w(x))$ is not defined for $w\in H^{-\varepsilon}$. Actually, it should be defined by $\int_I :P(w(x)):_{\mu_\lambda}dx$ where $:P(w(x)):$ denotes the Wick product. However this is not a smooth function on $W=H^{-\varepsilon}$ and cannot be covered by Theorem 3.2. This will be studied in [**3**].
(2) Let $H=H^2$. Then μ can be defined on $W=H^1$. For $0<\delta<1/2$, let $A=U(m^2-\Delta)^{\frac{1}{2}(\frac{1}{2}-\delta)}U^{-1}$, where U is the natural isometry from X to H. Let $Q(u)=\frac{1}{4}m^{1-2\delta}u^2+P(u)$, where $P(u)$ is the polynomial defined in (1). Let $\{c_1,\ldots,c_n\}$ be the minimum points of Q and asssume that $Q''(c_i)>0$ $(1\le i\le n)$. Again let $\tilde{V}(h)=\int_I P(h(x))dx$ for $h\in H$. Then we see that $\tilde{V}(h)-l\min Q$ can be extended to a smooth function $V(w)$ on W. Then the zero point set of $U(h)=\frac{1}{4}\|Ah\|_H^2+V(h)$ is the set of the constant functions $\{c_1,\ldots,c_n\}$. For this V and A, all assumptions in Theorem 3.2 hold with $\gamma_0=1+\frac{4\delta}{1-2\delta}$ and $\gamma=1+\gamma_0$.

We prove these theorems after preparations. Here we just prove $AK_iA \in L_1(H)$ under (A5). Since $V \in C^2(W)$, there exists a bounded linear operator $\hat{K}_i$ on W such that $D^2V(h_i)(u,v) = \left(\hat{K}_i u, v\right)_W$ for any $u, v \in W$. By the definition of the norm of W, there exists $\tilde{K}_i \in L(H)$ such that $\hat{K}_i = A^{\gamma}\tilde{K}_i A^{-\gamma}$. Thus for any $u, v \in H \subset W$,

$$D^2V(h_i)(u,v) = \left(\hat{K}_i u, v\right)_W = \left(A^{-\gamma}A^{\gamma}\tilde{K}_i A^{-\gamma}u, A^{-\gamma}v\right)_H = \left(A^{-\gamma}\tilde{K}_i A^{-\gamma}u, v\right)_H.$$

This shows $K_i = A^{-\gamma}\tilde{K}_i A^{-\gamma}$ and $AK_iA = A^{1-\gamma}\tilde{K}_i A^{1-\gamma}$. Because $\gamma - 1 \geq \gamma_0$, $A^{1-\gamma}$ is a Hilbert-Schmidt operator and this implies AK_iA is a trace class operator on H.

In our main theorems, we may assume that $c_A = 1$. Because, if Theorems hold in the case where $c_A = 1$, then it implies that $E_0\left(\lambda, \frac{A}{\sqrt{c_A}}, \frac{V}{c_A}\right) = e\left(\frac{A}{\sqrt{c_A}}, \frac{V}{c_A}\right)$. This shows the general cases.

The proof of upper bound is standard. Let χ be a smooth function on $\mathbb{R}$ satisfying $0 \leq \chi(x) \leq 1$, $\chi(x) = 1$ for $x \in [-1,1]$ and $\chi(x) = 0$ for $|x| \geq 2$. For $2/3 < \delta < 1$, set

$$\tilde{\Omega}_{\lambda,A,V_{K_i,h_i}}(w) = Z_\lambda \Omega_{\lambda,A,V_{K_i,h_i}}(w)\chi\left(\lambda^{\delta}\|w - h_i\|_W^2\right).$$

Here Z_λ is a normalization constant which makes the L^2-norm to be equal to 1. It holds that $\lim_{\lambda\to\infty} Z_\lambda = 1$. Since h_i is a minimizer of U, for any $k \in \mathrm{D}(A)$, $\frac{1}{2}\left(Ah_i, Ak\right)_H + DV(h_i)(k) = 0$. The fact $DV(h_i) \in H^*$ implies that $h_i \in \mathrm{D}(A^2)$ and $DV(h_i) = -\frac{1}{2}A^2h_i$. Using this and by the Taylor expansion, we have

$$\begin{aligned}
(3.2) \qquad V(w) &= V(h_i) + DV(h_i)(w - h_i) + (K_i(w-h_i), w - h_i) \\
&\quad + \frac{1}{3!}DV^3(w + \theta(w-h_i))((w-h_i)^{\otimes 3}) \\
&= \frac{1}{4}\|Ah_i\|_H^2 - \frac{1}{2}\left(A^2h_i, w\right) + (K_i(w-h_i), w-h_i) + R_{h_i}(w) \\
&= V_{K_i,h_i}(w) + R_{h_i}(w).
\end{aligned}$$

Here we denote the remainder term by $R_{h_i}(w)$. If $\chi(\lambda^{\delta}\|w - h_i\|_W^2) \neq 0$, then $|R_{h_i}(w)| \leq C\lambda^{-3\delta/2}$. This and the tail estimate of the Gaussian measure shows that

$$\mathcal{E}_{\lambda,A,V}\left(\tilde{\Omega}_{\lambda,A,V_{K_i,h_i}}, \tilde{\Omega}_{\lambda,A,V_{K_i,h_i}}\right) = E_0(\lambda, A, K_i) + O(\lambda^{2-\frac{3}{2}\delta}).$$

This proves the upper bound.

To prove the lower bound estimates, it suffices to prove the following Lemma 3.4. Let R be a sufficiently large positive number. Set $\chi_{i,R}(w) = \chi\left(R\|w - h_i\|_W^2\right)$ $(1 \leq i \leq n)$ and $\chi_{0,R}(w) = \sqrt{1 - \sum_{i=1}^n \chi_{i,R}(w)^2}$.

Lemma 3.4. — *Let us assume that the conditions of either Theorem* 3.1 *or Theorem* 3.2 *hold.*