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Pascal Auscher, Philippe Tchamitchian

Abstract. — We present in this work recent progress on the square root problem of Kato for differential operators in divergence form on $\mathbb{R}^n$. We discuss topics on functional calculus, heat and resolvent kernel estimates, square function estimates and Carleson measure estimates for square roots. In the first chapter, we show in a quantitative way how the theorems of Aronson-Nash and of De Giorgi are equivalent. In the central chapters, we take advantage of recent development in functional calculus and in harmonic analysis to propose a new point of view on Kato's problem which allows us to unify previous results and extend them. In the last chapter we study the associated Riesz transforms, their relation to Calderón-Zygmund operators and their behavior on L^p-spaces.

Résumé **(Problème de la racine carrée pour les opérateurs sous forme divergence et sujets connexes)**. — Ce travail a pour thème principal le problème de Kato concernant la racine carrée des opérateurs différentiels elliptiques sous forme divergence dans $\mathbb{R}^n$. Pour mener à bien cette étude, nous nous intéressons à des questions relatives au calcul fonctionnel, aux estimations de noyaux, aux fonctionnelles quadratiques et aux mesures de Carleson associées aux racines carrées. Dans le premier chapitre, nous montrons en un sens précis comment les théorèmes d'Aronson-Nash et de De Giorgi sont équivalents. Dans les deux chapitres centraux, nous tirons parti de développements récents sur le calcul fonctionnel et en analyse harmonique pour proposer un nouveau point de vue sur le problème de Kato qui permet d'unifier les résultats antérieurs et de les généraliser. Enfin, dans le dernier chapitre, nous étudions les transformées de Riesz associées, leur relation aux opérateurs de Calderón-Zygmund et leur comportement sur les espaces L^p.

Contents

INTRODUCTION

The main goal of this memoir is the study of square roots of divergence form differential operators $L = -\operatorname{div}(A\nabla)$ on $\mathbb{R}^n$, $n \geq 2$, where $A(x)$ is a matrix with complex-valued bounded entries and satisfying a uniform ellipticity condition. This operator is defined using the theory of maximal accretive operators on $L^2(\mathbb{R}^n)$ via the associated variational form on the Sobolev space $H^1(\mathbb{R}^n)$. We denote by $L^{1/2}$ the square root of L.

We are interested in the following questions.

Question 1. — *Does the domain of $L^{1/2}$ agree with $H^1(\mathbb{R}^n)$?*

When L is a selfadjoint operator or has smooth coefficients (*e.g.*, Lipschitz), this is true. However, the answer is not known in general.

Question 2. — *If A_0 is such that the domain of $L_0^{1/2}$ is $H^1(\mathbb{R}^n)$, where $L_0 = -\operatorname{div}(A_0\nabla)$, is the map $A \to L^{1/2}$, valued in $\mathcal{B}(H^1(\mathbb{R}^n), L^2(\mathbb{R}^n))$, complex analytic at A_0?*

Classical complex function theory tells us that analyticity of a Taylor expansion follows from boundedness in complex balls. This is the same principle that partly justifies the use of complex-valued coefficients here.

Question 3. — *If L is such that the domain of $L^{1/2}$ agrees with $H^1(\mathbb{R}^n)$, what can we say about L^p-boundedness, $p \neq 2$, of the Riesz transforms associated to L, namely $\nabla L^{-1/2}$? More generally, how do $\|L^{1/2}f\|_p$ and $\|\nabla f\|_p$ compare?*

In this work we bring new answers to the first two questions and, under a technical hypothesis that the kernel of the semigroup generated by $-L$ has Gaussian upper bounds and regularity, we completely elucidate the third one. Some results were announced in [**15**].