Astérisque

RAINER WEISSAUER

# Four dimensional Galois representations

*Astérisque*, tome 302 (2005), p. 67-150
<http://www.numdam.org/item?id=AST_2005__302__67_0>

*Astérisque*
**302**, 2005, p. 67–150

# FOUR DIMENSIONAL GALOIS REPRESENTATIONS

*by*

Rainer Weissauer

***Abstract.*** — We construct four dimensional irreducible mixed $l$-adic representations of the absolute Galois group of $\mathbb{Q}$, which are attached to irreducible cuspidal automorphic representations $\Pi$ of the symplectic group of similitudes $\mathrm{GSp}(4)$, whose archimedean component $\Pi_\infty$ belongs to the discrete series, and discuss some of the properties of these $l$-adic representations.

***Résumé*** **(Représentations galoisiennes de dimension quatre).** — Nous construisons et étudions certaines représentations $l$-adiques mixtes irréductibles de dimension quatre du groupe de Galois absolu de $\mathbb{Q}$, attachées à des representations automorphes cuspidales irréductibles $\Pi$ du groupe de similitudes symplectiques $\mathrm{GSp}(4)$, dont la composante archimédienne $\Pi_\infty$ appartient à la série discrète. Nous présentons également quelques propriétés de ces représentations $l$-adiques.

## Introduction

It is well known how to associate two dimensional $\lambda$-adic representations to irreducible cuspidal automorphic representations of the group $\mathrm{Gl}(2,\mathbb{A})$, whose archimedian component is a discrete series representation, for the ring $\mathbb{A}$ of rational adeles. In the case $\mathrm{Gl}(2)$ the condition at the archimedian place leads to the study of classical holomorphic cuspforms of weight $k \geqslant 2$. Already for the symplectic group of similitudes $\mathrm{GSp}(4,\mathbb{A})$ the corresponding situation is not understood as well. In this paper we derive analogous results for the group $\mathrm{GSp}(4)$ by constructing corresponding four dimensional Galois representations. Furthermore we discuss various properties of these representations. Proofs are based on certain fundamental assertions, in particular from spectral theory, which itself will not be discussed in this paper. Some of them are available only in preprint form. For the convenience of the reader they will here be formulated as hypotheses, in order to make the paper self contained.

**2000** ***Mathematics Subject Classification.*** — 11F80, 11F72, 11G18, 11G40.
***Key words and phrases.*** — Shimura varieties, Galois representations.

That for an irreducible cuspidal automorphic representation $\Pi$ of $\mathrm{GSp}(4,\mathbb{A})$ the representation $\Pi_\infty$ at the archimedian place belongs to the discrete series, does not necessarily lead to the study of holomorphic cuspforms. The reason is, that the discrete series representations of $\mathrm{GSp}(4,\mathbb{R})$ are parameterized by $L$-packets. Each $L$-packet contains two classes of irreducible representations. One of them is a member of the holomorphic discrete series and does not have a Whittaker model, whereas the other is nonholomorphic but has a Whittaker model. The packets itself are parameterized, up to a character twist, by what we call their weight. The weight is described by a pair of integers $(k_1,k_2)$ such that $k_1 \geqslant k_2 \geqslant 3$. An irreducible, cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$, whose archimedian component is holomorphic of weight $(k_1,k_2)$, corresponds to classical vector valued holomorphic Siegel modular forms $f(\Omega)$ on the Siegel upper half space $H$ of genus two $f : H \to V_\rho$ with the transformation property

$$f((A\Omega+B)(C\Omega+D)^{-1}) = \rho(C\Omega+D)\cdot f(\Omega), \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the representation $\rho = \mathrm{Symm}^{k_1-k_2}\otimes \det^{k_2}$ of $\mathrm{Gl}(2,\mathbb{C})$ on $V_\rho$, where $M$ is in a congruence subgroup of the Siegel modular group. In the case $k=k_1=k_2$ we obtain classical Siegel modular forms of weight $k$. The lowest $K_\infty$-type of the holomorphic discrete series representation is characterized by its highest weight vector, which is defined by the weight $(k_1,k_2)$. In the Whittaker case the corresponding $K_\infty$-type has highest weight $(k_1, 2-k_2)$.

Let $\mathbb{A} = \mathbb{R}\times\mathbb{A}_f$ be the ring of rational adeles. In the following let $\Pi = \Pi_\infty \Pi_f$ be an irreducible cuspidal automorphic representation of the group $\mathrm{GSp}(4,\mathbb{A})$, whose component $\Pi_\infty$ belongs to the discrete series lying in a $L$-packet of weight $(k_1,k_2)$. We abbreviate this by saying, that $\Pi$ has weight $(k_1,k_2)$. The ramified places of $\Pi$ are the archimedian place and the finite places, where $\Pi$ is not spherical. The first result is

***Theorem I***. — *Suppose $\Pi$ is a unitary cuspidal irreducible automorphic representation of $\mathrm{GSp}(4,\mathbb{A})$ for which $\Pi_\infty$ belongs to the discrete series of weight $(k_1,k_2)$. Let $S$ denote the set of ramified places of the representation $\Pi$. Put $w = k_1+k_2-3$. Then there exists a number field $E$, such that for primes $p\notin S$ the local $L$-factor*

$$L_p(p^{-s}) = L_p(\Pi_p, s-w/2), \quad L_p(X)^{-1} \in E[X]$$

*of the degree 4 spinor $L$-series (for the 'algebraic' normalization involving the shift by $-w/2$ as above) has coefficients in $E$, and such that for any prime number $l$ and any extension $\lambda$ of $l$ to $E$ there exists a four dimensional semisimple Galois representation*

$$\rho_{\Pi,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}:\mathbb{Q}) \longrightarrow \mathrm{Gl}(4,\overline{E_\lambda}),$$

*which is unramified outside $S\cup\{l\}$, so that for $p\notin S\cup\{l\}$ the following holds*

$$L_p(\Pi_p, s-w/2) = \det(1-\rho_{\Pi,\lambda}(\mathrm{Frob}_p)p^{-s})^{-1}.$$

*The eigenvalues of* $\rho_{\Pi,\lambda}(\mathrm{Frob}_p)$ *for* $p \neq l, p \notin S$ *are algebraic integers. The representation* $\rho_{\Pi,\lambda}$ *arises from a* $\lambda$*-adic representation, if* $E$ *is chosen large enough. The so defined* $\lambda$*-adic representation* $\rho_{\Pi,\lambda}$ *is mixed. If* $\Pi$ *is not a CAP representation (for this notation see* [**S**]*) the representation* $\rho_{\Pi,\lambda}$ *is pure of weight* $w$*, i.e. for all isomorphisms* $\overline{\mathbb{Q}}_l \cong \mathbb{C}$ *the image of the eigenvalues of* $\rho_{\Pi,\lambda}(\mathrm{Frob}_p)$ *has absolute value* $p^{w/2}$ *for* $p \neq l$ *and* $p \notin S$.

***1. Remark.*** — $\mathrm{Frob}_p$ is the geometric Frobenius. The chosen normalization of the $\lambda$-adic representation $\rho_{\Pi,\Lambda}$ is of cohomological nature. In the most relevant cases the formulas above first arise from a right action of the Galois group on certain cohomology groups. In order to obtain a representation in the usual sense one has to consider the transposed, which then defines a representation in the usual sense. Since characteristic polynomials do not change under transposition this does not matter. Nevertheless it is the dual of the representation so obtained, which corresponds to what usually appears in the literature on elliptic modular forms. The dual representation $\rho_{\Pi,\lambda}^{\vee}$ is

$$\rho_{\Pi,\lambda}^{\vee} \cong \rho_{\Pi,\lambda} \otimes \chi^{-1}, \quad \chi = \omega_{\Pi} \cdot \mu_l^{-w},$$

where $\mu_l$ is the cyclotomic character $\mu_l(\mathrm{Frob}_p) = p^{-1}$ and $\omega_{\Pi}(\mathrm{Frob}_p) = \omega_{\Pi}(p)$, where $\omega_{\Pi}$ is the central character of $\Pi$. This is a consequence of the Tchebotarev density theorem, since the formulas $L_p(\Pi_p, s - w/2) = \det(1 - \rho_{\Pi,\lambda}(\mathrm{Frob}_p)p^{-s})^{-1}$ and $\Pi \cong \Pi^{\vee} \otimes \omega_{\Pi}$ imply that the two semisimple representations $\rho_{\Pi,\lambda}^{\vee}$ and $\rho_{\Pi,\lambda} \otimes \chi^{-1}$ (with $\chi = \omega_{\Pi} \cdot \mu_l^{-w}$) have the same character. Similarly, the class of the representation $\rho_{\Pi,\lambda}$ only depends on the weak equivalence class of $\Pi$. Two irreducible automorphic representations $\Pi_1, \Pi_2$ are called weakly equivalent, if they are isomorphic locally $\Pi_{1,v} \cong \Pi_{2,v}$ at almost all places $v$.

***Theorem II.*** — *The representations* $\rho_{\Pi,\lambda}$ *constructed in theorem I are never reducible of the form* $\rho_{\Pi,\lambda} \cong \rho_0 \oplus \rho_0$*, for a two-dimensional* $\lambda$*-adic representations* $\rho_0$ *of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*. They contain a one dimensional invariant subspace if and only if* $\Pi$ *is a CAP-representation (of Saito-Kurokawa type* [**P**]*).*

Suppose $E$ is chosen large enough, so that the representation $\rho_{\Pi,\lambda}$ is defined over $E_\lambda$. Then $\rho_{\Pi,\lambda}$ can be viewed as representations of dimension $4 \cdot [E_\lambda : \mathbb{Q}_l]$ over $\mathbb{Q}_l$. But in fact, by the way they will be constructed, these $\mathbb{Q}_l$-vector spaces then turn out to be Hodge-Tate modules of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$ using [**CF**] theorem 6.2. Moreover if we exclude certain exceptional cases — for the notion of a weak endoscopic lift see the definition further below — we have

***Theorem III.*** — *Suppose the cuspidal representation* $\Pi$ *is neither CAP nor a weak endoscopic lift and weakly equivalent to a multiplicity one representation. Then the* *representations $\rho_{\Pi,\lambda}$ define Hodge-Tate modules over $\mathbb{Q}_l$ with four different Hodge types*

$$(k_1 + k_2 - 3, 0),\ (k_1 - 1, k_2 - 2),\ (k_2 - 2, k_1 - 1),\ (0, k_1 + k_2 - 3)$$

*each of which occurs with the same $\mathbb{Q}_l$-dimension $[E_\lambda : \mathbb{Q}_l]$.*

Theorem III is deduced from proposition 1.5.

***2. Remark***. — A weaker version of theorem I was obtained in [**T**]. As in [**T**], p. 291ff we use the fact, that the representation $\Pi$ contributes to the (interior) cohomology of a suitably defined projective limit $M$ of Siegel modular threefolds with respect to a coefficient system $\mathcal{V}_\mu(\overline{\mathbb{Q}}_l)$, which only depends on the weight $(k_1, k_2)$ of $\Pi$. Our result is also deduced from the study of the etale cohomology groups. That the etale cohomology of $M$ defines mixed Galois representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, is seen by discussing three cases separately: the case of CAP-representations, the case of weak endoscopic lifts and the remaining case. In the last case the representations $\rho_{\Pi,\lambda}$ defined above do naturally occur in the third cohomology of $M$. This fails to hold in the case, where $\Pi$ is a CAP-representation of Saito-Kurokawa type. It also fails to hold in the case, where $\Pi$ is a weak endoscopic lift considered below. Both these exceptional cases are interesting for various reasons.

***Definition***. — A unitary irreducible cuspidal representation $\Pi$ of $\mathrm{GSp}(4, \mathbb{A})$ is called a *weak endoscopic lift*, if there exist two unitary irreducible cuspidal automorphic forms $\pi_1, \pi_2$ of $\mathrm{Gl}(2, \mathbb{A})$ with central characters $\omega_{\pi_1} = \omega_{\pi_2}$, such that

$$L_v(\Pi, s) = L_v(\pi_1, s) L_v(\pi_2, s)$$

holds for almost all places. Here $L_v(\Pi, s)$ denotes the local $L$-factor of the degree 4 spinor $L$-series.

Let $\Pi$ be a weak endoscopic lift attached to $\pi_1, \pi_2$. Then under the hypothesis A formulated in the next paragraph we get $\omega_{\pi_i} = \omega_\Pi$. Furthermore, if we consider representations $\Pi$ for which $\Pi_\infty$ belongs to the discrete series, $\pi_{\infty,i}$ will belong to the discrete series of weight $r_i$, such that $r_1 > r_2 \geqslant 2$ holds for a suitable ordering. Conversely for $\pi_1, \pi_2$ with archimedian components as above $\sigma = (\pi_1, \pi_2)$ lifts to a global nontrivial $L$-packet, defined as the weak equivalence class of unitary cuspidal irreducible automorphic representations $\Pi$ of $\mathrm{GSp}(4, \mathbb{A})$, whose components at infinity belong to the discrete series of weight $(k_1, k_2)$ such that the $L$-identities (1) from above hold at almost all places. The integers $k_i$ and $r_i$ are related by the formulas $r_1 = k_1 + k_2 - 2$ and $r_2 = k_1 - k_2 + 2$.

***Hypothesis A***. — Let $F$ be a totally real number field. Let $\sigma = (\pi_1, \pi_2)$ be a pair of irreducible cuspidal representations of $\mathrm{Gl}(2, \mathbb{A}_F)$ with a common central character