Bull. Soc. Math. France
146 (4), 2018, p. 675-690

ON ERGODIC AVERAGES FOR PARABOLIC PRODUCT FLOWS

by Alexander I. Bufetov & Boris Solomyak

Abstract. — We consider a direct product of a suspension flow over a substitution dynamical system and an arbitrary ergodic flow and give quantitative estimates for the speed of convergence for ergodic integrals of such systems. Our argument relies on new uniform estimates of the spectral measure for suspension flows over substitution dynamical systems. The paper answers a question by Jon Chaika.

Résumé (*Les moyennes ergodiques des produits cartésiens des flots paraboliques*). — Pour le produit cartésien d'un flot ergodique arbitraire avec un flot de suspension sur un système de substitution, nous estimons la vitesse de convergence des intégrales ergodiques. Notre argument se base sur les bornes uniformes pour les mesures spectrales des flots de suspension sur les systèmes de substitution. Notre résultat répond à une question de Jon Chaika.

Texte reçu le 29 décembre 2016, modifié le 3 mai 2017, accepté le 1er juin 2017.

Alexander I. Bufetov, Aix-Marseille Université, CNRS Centrale Marseille, I2M, UMR 7373 39 rue F. Joliot Curie Marseille France, Steklov Mathematical Institute of RAS Moscow Russia, Institute for Information Transmission Problems Moscow Russia, National Research University Higher School of Economics Moscow Russia, The Chebyshev Laboratory Saint-Petersburg State University Saint Petersburg Russia • *E-mail :* alexander.bufetov@univ-amu.fr,bufetov@mi.ras.ru

Boris Solomyak, Bar-Ilan University Department of Mathematics Ramat-Gan Israel • *E-mail :* bsolom3@gmail.com

Mathematical subject classification (2010). — 37A30, 37B10.

Key words and phrases. — Substitution dynamical system, spectral measure, Hölder continuity.

doi:10.24033/bsmf2770

1. Introduction

Parabolic dynamical systems are characterized by a “slow” chaotic behavior: whereas for hyperbolic systems nearby trajectories diverge exponentially, for parabolic ones they diverge polynomially in time. Classical examples include the horocycle flows and translation flows on flat surfaces of higher genus. Substitution dynamical systems and suspension flows over them also fall into this category. Due to their simple-to-describe combinatorial framework and many connections, e.g. with number theory and automata theory, they have provided a “testing ground” for new methods. Their spectral theory has been actively studied, but many natural questions remain open.

We refer the reader to [12, 9] for a detailed background, but recall the basic definitions briefly. Let $\mathcal{A} = \{1, \dots, m\}$ be a finite alphabet; we denote by $\mathcal{A}^+$ the set of finite (non-empty) words in $\mathcal{A}$. A *substitution* is a map $\zeta : \mathcal{A} \to \mathcal{A}^+$, which is extended to an action on $\mathcal{A}^+$ and $\mathcal{A}^{\mathbb{N}}$ by concatenation. (Using different language, this is a morphism of a free semigroup with $\mathcal{A}$ being a set of free generators.) The *substitution space*, denoted X_ζ, is a subset of $\mathcal{A}^{\mathbb{Z}}$ consisting of all two-sided infinite sequences x with the property that for every $n \in N$, the word, or block, $x[-n, n]$ occurs as a subword in $\zeta^k(a)$ for some $k \in \mathbb{N}$ and $a \in \mathcal{A}$. It is clearly closed (in the discrete product topology) and shift-invariant; thus we obtain a topological *substitution dynamical system* (X_ζ, T_ζ), where T_ζ denotes the left shift restricted to X_ζ. The *substitution matrix* is defined by

$$\mathsf{S}_\zeta(i, j) = \text{number of symbols } i \text{ in the word } \zeta(j).$$

This is a non-negative integer $m \times m$ matrix, which provides the *abelianization* of the free semigroup morphism ζ. Assume that ζ is *primitive*, that is, some power of S_ζ has only positive entries. In this case the $\mathbb{Z}$-action (X_ζ, T_ζ) is minimal and uniquely ergodic, with a unique invariant Borel probability measure μ. We also assume that ζ is *aperiodic*, i.e. the system has no periodic points, excluding the trivial case of X_ζ finite. Denote by θ_j, $j \geq 1$, the eigenvalues of S_ζ ordered by magnitude:

$$\theta_1 > |\theta_2| \geq \dots$$

The famous “Pisot substitution conjecture” asserts that if $|\theta_2| < 1$, then the measure-preserving system has a pure discrete spectrum. (This condition is equivalent to θ_1 being a Pisot number and the characteristic polynomial of S_ζ being irreducible.) This is known only in the two-symbol case [4, 10], although there has been a lot of progress recently, see [2]. In any case, such substitution systems have a large discrete component: they have a factor which is an irrational translation on an $(m-1)$-dimensional torus.

Along with the substitution $\mathbb{Z}$-action, it is natural to study suspension flows over them. We consider only piecewise-constant roof functions. More precisely, for a strictly positive vector $\vec{s} = (s_1, \dots, s_m)$ we consider the suspension flow over T_ζ, with the piecewise-constant roof function, equal to s_j on the cylinder set $[j]$. The resulting space will be denoted by $\mathfrak{X}_\zeta^{\vec{s}}$, the unique invariant measure for our suspension flow by $\widetilde{\mu}$ and the flow by $(\mathfrak{X}_\zeta^{\vec{s}}, \widetilde{\mu}, h_t)$. We have, by definition,

$$\mathfrak{X}_\zeta^{\vec{s}} = \bigcup_{a\in\mathcal{A}} \mathfrak{X}_a, \quad \text{where } \mathfrak{X}_a = \{(x,t):\ x\in X_\zeta,\ x_0=a,\ 0\le t\le s_a\}$$

and this union is disjoint in measure. We call the system $(\mathfrak{X}_\zeta^{\vec{s}}, \widetilde{\mu}, h_t)$ a *substitution* $\mathbb{R}$*-action*. This flow can also be viewed as the translation action on a tiling space, with interval prototiles of length s_j. A special case of interest is when $\vec{s}$ is the Perron-Frobenius eigenvector for the transpose substitution matrix S_ζ^t; this corresponds to the self-similar tiling on the line, see [14].

Sometimes results on spectral properties become simpler when we pass from the $\mathbb{Z}$-action to the $\mathbb{R}$-action. In particular, the condition "θ_1 is not Pisot" is equivalent to the substitution $\mathbb{R}$-action being weakly mixing, in the self-similar case [14] and in the "generic case" (for Lebesgue-a.e. $\vec{s}$) [6], whereas the situation for substitution $\mathbb{Z}$-actions is much more complicated [13, 8]. In this paper we continue the analysis of the generically weak-mixing case, assuming $|\theta_2|>1$, which was started in [5]. Note that the "borderline" case $|\theta_2|=1$ is more subtle [3]. We should also note that when the characteristic polynomial of S_ζ is reducible, e.g. when θ_1 is an integer, the type of spectrum is determined not just by the matrix, but also by the order of the letters in the words $\zeta(j)$, see e.g. [12].

As is well known, weak mixing of a system is equivalent to the ergodicity of the product flow $h_t\times H_t$, where H_t is an arbitrary measure-preserving ergodic flow defined on a standard probability space (Y,ν). We thus have

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R \langle (f\otimes g)\circ(h_t\times H_t), f\otimes g\rangle dt = 0 \qquad (1)$$
$$\text{for all } f\in L^2(\mathfrak{X}_\zeta^{\vec{s}},\widetilde{\mu}), g\in L_0^2(Y,\nu).$$

Our aim in this paper is to give power estimates for the speed of convergence in (1).

On $\mathfrak{X}_\zeta^{\vec{s}}$ we consider Lipschitz "cylindrical functions", namely, functions of the form

$$f(x,t)=\psi_{x_0},\quad 0\le t\le s_{x_0},$$

where $\psi_j\in \mathrm{Lip}[0,s_j]$, $j\le m$. Denote

$$\|f\|_L := \max_{a\in\mathcal{A}}\|\psi_a\|_L,$$

where $\|\cdot\|$ is the Lipschitz norm.

Let $f \in L^2(\mathfrak{X}_\zeta^{\vec{s}}, \widetilde{\mu})$. By the spectral theorem for measure-preserving flows, there is a finite positive Borel measure σ_f on $\mathbb{R}$ such that

$$\widehat{\sigma}_f(-t) = \int_{-\infty}^{\infty} e^{2\pi i\omega t} d\sigma_f(\omega) = \langle f \circ h_t, f\rangle \quad \text{for } t \in \mathbb{R},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in L^2. For functions f and g set $(f \otimes g)(x, y) := f(x)g(y)$. Without loss of generality, we can restrict ourselves to roof vectors from the simplex $\Delta^{m-1} = \{\vec{s} \in \mathbb{R}_+^m : \sum_{j=1}^m s_j = 1\}$.

Theorem 1.1. — *Let ζ be a primitive aperiodic substitution on $\mathcal{A} = \{1, \ldots, m\}$, with substitution matrix S. Suppose that the characteristic polynomial of S is irreducible and the second eigenvalue satisfies $|\theta_2| > 1$. Then there exists a constant $\alpha > 0$, depending only on the substitution ζ, such that for Lebesgue-almost every $\vec{s} \in \Delta^{m-1}$, for every Lip-cylindrical function f, with $\int f d\widetilde{\mu} = 0$, and for any ergodic flow (Y, H_t, ν) and a function $g \in L_0^2(Y, \nu)$,*

$$\left| \int_0^R \langle (f \otimes g) \circ (h_t \times H_t), f \otimes g\rangle dt \right| \leq CR^{1-\alpha}, \quad R > 0, \tag{2}$$

where $C = C(\vec{s}, \|f\|_L, \|g\|_2) > 0$.

In order to prove Theorem 1.1, we need the following strengthening of Theorem 4.1 in [5].

Theorem 1.2. — *Let ζ be a primitive aperiodic substitution on $\mathcal{A}$, satisfying the assumptions of Theorem 1.1. Then there exists a constant $\gamma > 0$, depending only on the substitution ζ, such that for Lebesgue-almost every $\vec{s} \in \Delta^{m-1}$ there exists $r_0 = r_0(\vec{s}) > 0$, such that for every Lip-cylindrical function f, with $\int f d\widetilde{\mu} = 0$,*

$$\sigma_f([\omega - r, \omega + r]) \leq Cr^\gamma, \quad \text{for all } \omega \in \mathbb{R} \text{ and } 0 < r \leq r_0. \tag{3}$$

Here, σ_f is the spectral measure of f corresponding to the suspension flow $(\mathfrak{X}_\zeta^{\vec{s}}, h_t)$ and $C > 0$ depends only on $\|f\|_L$.

The improvement upon theorem 4.1 in [5] is that in (3), our local Hölder estimates are **uniform** on the **whole line**, while in [5] we were able to prove our estimates to be uniform only away from zero and infinity. An estimate of the Hausdorff dimension of the exceptional set of suspension flows can also be given, cf. [5, Theorem 4.2].

2. Proof of Theorem 1.1 assuming Theorem 1.2

By a result of Strichartz [15, Cor. 5.2] we immediately obtain from (3):

$$\sup_y \sup_{R\geq 1} R^{\gamma-1} \int_{y-R}^{y+R} |\widehat{\sigma}_f(\zeta)|^2 d\zeta \leq C. \tag{4}$$

By the definition of spectral measures,

$$\int_0^R \widehat{\sigma}_f(-\zeta)\widehat{\sigma}_g(-\zeta)d\zeta = \int_0^R \left(\int_{\mathfrak{X}_\zeta^{\vec{s}}} (f\circ h_t)\overline{f}d\widetilde{\mu}\right)\left(\int_Y (g\circ H_t)\overline{g}d\nu\right)$$
$$= \int_0^R \int_{\mathfrak{X}_\zeta^{\vec{s}}\times Y} ((f\otimes g)\circ(h_t\times H_t))\,(\overline{f\otimes g})d(\widetilde{\mu}\times\nu),$$

which is exactly the expression in (2) under the absolute value sign. It remains to note that

$$\left|\int_0^R \widehat{\sigma}_f(-\zeta)\widehat{\sigma}_g(-\zeta)d\zeta\right| \leq \left(\int_0^R |\widehat{\sigma}_f(-\zeta)|^2 d\zeta\right)^{1/2}\left(\int_0^R |\widehat{\sigma}_g(-\zeta)|^2 d\zeta\right)^{1/2}$$
$$\leq C^{1/2}R^{1-\frac{\gamma}{2}}\|g\|_2,$$

applying Cauchy–Bunyakovsky–Schwarz, (4), and the simple bound $\|\widehat{\sigma}_g\|_\infty \leq \|g\|_2^2$. □

The plan of the proof of Theorem 1.2 is as follows: we go through the proof of [5, Theorem 4.2], making it more quantitative, and obtain

Proposition 2.1. — *Let ζ be a primitive aperiodic substitution on $\mathcal{A}$, satisfying the assumptions of Theorem 1.1. Then there exist constants $\widetilde{\gamma}, Z > 0$, depending only on the substitution ζ, such that for Lebesgue-almost every $\vec{s} \in \Delta^{m-1}$ there exists $r_0 = r_0(\vec{s})$, such that for every Lip-cylindrical function f and $\omega \neq 0$,*

$$\sigma_f([\omega - r, \omega + r]) \leq C\cdot r^{\widetilde{\gamma}}, \quad \text{for } 0 < r < r_0|\omega|^Z. \tag{5}$$

Here, the constant $C = C(\|f\|_L) > 0$ depends only on the Lip-norm of f.

Note that here we do not have to assume $\int_X f d\widetilde{\mu} = 0$. We will then "glue" this Hölder bound with the Hölder bound at $\omega = 0$ (which essentially follows from a result of Adamczewski [1]) in the case when f has mean zero.

3. Twisted ergodic integrals and spectral measures

Let (Y, μ, h_y) be a measure-preserving flow. For $f \in L^2(Y,\mu)$, $R > 0, \omega \in \mathbb{R}$, and $y \in Y$ consider the "twisted Birkhoff integral"

$$S_R^{(y)}(f,\omega) = \int_0^R e^{-2\pi i\omega t} f(h_t y)dt.$$

Recall the following standard lemma; a proof may be found in [5, Lemma 4.3].