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BIRATIONAL GEOMETRY OF FOLIATIONS ASSOCIATED TO SIMPLE DERIVATIONS

by Gaël Cousin, Luís Gustavo Mendes & Iván Pan

Abstract. — We propose a study of the foliations of the projective plane induced by simple derivations of the polynomial ring in two indeterminates over the complex field. These correspond to foliations which have no invariant algebraic curve nor singularities in the complement of a line. We establish the position of these foliations in the birational classification of foliations and prove the finiteness of their birational symmetries. Most of the results apply to wider classes of foliations.

Résumé (*Géométrie birationnelle des feuilletages associés à des dérivations simples*). — Nous proposons une étude des feuilletages du plan projectif qui sont induits par une dérivation simple de l'anneau des polynômes à deux indéterminées sur le corps des complexes. Il s'agit des feuilletages qui ne possèdent ni courbe algébrique ni singularité dans le complémentaire d'une droite. Nous donnons leur emplacement dans la classification birationnelle des feuilletages et prouvons la finitude de leurs groupes de symétries birationnelles. La plupart des résultats s'applique en fait à des classes plus larges de feuilletages.

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Gaël Cousin, GMA-IME-UFF, Campus do Gragoatá, Niterói, RJ, Brazil • *E-mail :* gcousin@id.uff.br

Luís Gustavo Mendes, Instituto de Matemática, UFRGS, Porto Alegre, Brazil • *E-mail :* mendes@mat.ufrgs.br

Iván Pan, Centro de Matemática, Facultad de Ciencias, UdelaR, Montevideo, Uruguay • *E-mail :* ivan@cmat.edu.uy

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1. Introduction and results

> "(. . .) mais nous ne serons satisfaits que quand on aura trouvé un certain groupe de transformations (par exemple de transformations de Cremona) qui jouera, par rapport aux équations différentielles, le même rôle que le groupe des transformations birationnelles pour les courbes algébriques. Nous pourrons alors ranger dans une même classe toutes les transformées d'une même equation."
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> — Henri Poincaré, L'Avenir des Mathématiques

At least since Poincaré [32], the study of the algebraic subsets that are left invariant by a given plane polynomial vector field is known to be a difficult matter. For instance, the example $rx\partial_x + y\partial_y, r \in \mathbb{Q}$ shows that it is in general impossible to bound the degree of the invariant algebraic curves in terms of the degrees of the vector field's coefficients. In commutative algebra, the vector fields that preserve no nontrivial algebraic subset of the affine plane correspond to the so-called simple derivations and there is an extensive literature dedicated to the production of such examples, see [1, 14, 16, 17, 19, 22, 27, 28, 34], among others.

This is also an active field of study in foliation theory, where one considers the extension of the foliation to the projective plane. A key result in this context is the work of Jouanolou [18] that exhibited a family of examples of foliations without any invariant algebraic curve in the projective plane and deduced the (Baire) genericity of such examples. Compare [21] for a generalization. In the opposite direction, the study of foliations in the neighborhood of invariant divisors provides important information on the foliation [11] and the study of such divisors remains crucial for the study of algebraic or Liouvillian integrability of foliations [12].

At the turn of the century, the birational geometry of foliations had been developed and one has a birational classification *à la* Enriques-Kodaira for foliations of projective surfaces (see [4, 23, 24]). The goal of this article is to explain how the recent tools in foliation theory allow to classify geometrically the simple derivations of $\mathbb{C}[x, y]$ and to study their symmetries. We will also present a set of examples found throughout the commutative algebra literature and study their relationships.

In algebra, a derivation of the ring $\mathbb{C}[x, y]$ is said to be *simple* if it does not globally fix any nontrivial proper ideal. It corresponds to a polynomial vector field of $\mathbb{C}^2$ without zeroes and without algebraic trajectories.

For any derivation ∂, the isotropy group $\mathrm{Aut}(\partial)$ is composed of the $\mathbb{C}$-automorphisms $\rho : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ which verify

$$\rho\partial = \partial\rho.$$

Although there exists derivations with infinite isotropy group, the main result of [25] is that $\mathrm{Aut}(\partial)$ *is trivial for any simple derivation.*

Take $\rho \in \mathrm{Aut}(\mathbb{C}[x,y])$, $R : \mathbb{C}^2 \to \mathbb{C}^2$ the polynomial automorphism associated to ρ and let ω_∂ be the dual 1-form to the vector field $\partial = f\,\partial_x + g\,\partial_y$ (i.e. $\omega_\partial = g\,dx - f\,dy$). Then $\rho\partial = \partial\rho$ is equivalent to

$$R^*(\omega_\partial) = \mathrm{Jac}(R) \cdot \omega_\partial$$

where $\mathrm{Jac}(R) \in \mathbb{C}^*$ is the Jacobian determinant of R. A less restrictive condition is that

$$R^*(\omega_\partial) = c \cdot \omega_\partial,$$

for some $c \in \mathbb{C}^*$ (depending on R). This means that R preserves the *foliation* $\mathcal{F}_\partial$ of $\mathbb{C}^2$ associated to ∂ (or to ω_∂), see Remark 2.1 p. 612.

We denote by $\mathrm{Pol}(\mathcal{F}_\partial)$ the group consisting of polynomial automorphisms of $\mathbb{C}^2$ which preserve the foliation $\mathcal{F}_\partial$. There is a natural homomorphism

$$\mathrm{Aut}(\partial) \hookrightarrow \mathrm{Pol}(\mathcal{F}_\partial).$$

Let us denote by $\mathcal{F}$ the singular holomorphic foliation of the projective plane $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty$ which is the extension of $\mathcal{F}_\partial$ in $\mathbb{C}^2$. All along the paper, if ∂ is a simple derivation, both $\mathcal{F}_\partial$ in $\mathbb{C}^2$ and its extension $\mathcal{F}$ in $\mathbb{P}^2$ are called *foliations associated to simple derivations.*

The reader must be warned that, even if $\mathcal{F}_\partial$ has no singularity, some singularities of $\mathcal{F}$ along the line at infinity L_∞ are unavoidable, see [4, Prop. 2.1]. Also be aware that the line at infinity L_∞ may be invariant by $\mathcal{F}$.

Denote by $\mathrm{Bir}(\mathcal{F})$ the group of birational transformations of $\mathbb{P}^2$ which preserve a foliation $\mathcal{F}$; the elements in $\mathrm{Bir}(\mathcal{F})$ are sometimes called *birational symmetries* of $\mathcal{F}$. If $\mathcal{F}$ extends a foliation $\mathcal{F}_\partial$ of $\mathbb{C}^2$, then there is a natural homomorphism

$$\mathrm{Pol}(\mathcal{F}_\partial) \hookrightarrow \mathrm{Bir}(\mathcal{F})$$

whose meaning is that a (non-linear) polynomial automorphism of $\mathbb{C}^2$ extends to a special type of birational map of $\mathbb{P}^2$. Namely, a birational map with a unique (proper) point of indeterminacy $p \in L_\infty$, whose net effect on $\mathbb{P}^2$ is to replace L_∞ by the strict transform of the last exceptional curve introduced in the elimination of the indeterminacy point.

In Section 7, we propose a construction of simple derivations ∂ with arbitrary large finite $\mathrm{Pol}(\mathcal{F}_\partial)$. This shows the optimality of the following.

Theorem A. — *Let $\mathcal{F}$ be a foliation of $\mathbb{P}^2$ whose restriction $\mathcal{F}|_{\mathbb{C}^2}$ to $\mathbb{C}^2$ has no algebraic invariant curve. Then* $\mathrm{Bir}(\mathcal{F})$ *is finite; in particular, a foliation associated to a simple derivation admits only finitely many birational symmetries.*

Theorem A is actually derived from the next result which determines, in particular, the positions that foliations associated to simple derivations may occupy in the birational classification of foliations. This classification is based on the notion of the *Kodaira dimension* of a foliation, denoted $\kappa(\mathcal{F})$, whose range is $\kappa(\mathcal{F}) \in \{-\infty, 0, 1, 2\}$, see Section 2.

For a reduced divisor in a quasiprojective surface, we say it is a *rational* curve if its projective closure has geometric genus zero. By a *Riccati foliation* on $\mathbb{P}^2$ we mean a foliation which, up to a birational modification of $\mathbb{P}^2$, is everywhere transverse to the general fiber of a rational fibration (see § 2.13 for more details).

THEOREM B. — *Let $\mathcal{F}$ be a foliation of the projective plane such that the restriction $\mathcal{F}|_{\mathbb{C}^2}$ has no invariant rational curve.*

(i) Then $\kappa(\mathcal{F}) \geq 1$;

(ii) If $\mathcal{F}|_{\mathbb{C}^2}$ has no invariant algebraic curve, then $\kappa(\mathcal{F}) = 1$ if and only if $\mathcal{F}$ is a Riccati foliation.

(iii) The cases $\kappa(\mathcal{F}) \in \{1, 2\}$ are realized by foliations associated to simple derivations.

Note that Theorem B applies to a class of foliations which is larger than the one of foliations associated to simple derivations and that case B-(ii) includes the foliations associated to *Shamsuddin derivations cf.* [34].

In fact, we obtain Theorem B-(i) as a special case of the following with $(X, D) = (\mathbb{P}^2, L_\infty)$.

THEOREM C. — *Let X be a smooth projective rational surface and let D be a reduced divisor on X. Suppose that $X \setminus D$ is simply connected. Then any foliation $\mathcal{F}$ on X that possesses no invariant rational curve outside D satisfies $\kappa(\mathcal{F}) \geq 1$.*

In Section 5 we study the foliations associated to examples of simple derivations found throughout the literature and discuss their birational equivalence.

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2. Preliminaries on foliations

The paper relies on concepts and results of the theory of singularities and birational geometry of foliations on algebraic complex surfaces. We present some basic facts in this preliminary section but in the paper, when necessary, we refer the reader to the corresponding sections of [4] or [3], where the theory is masterfully explained.

2.1. First definitions. — On a smooth complex surface X, a foliation $\mathcal{F}$ is given by an open covering (U_i) of X and local vector fields $v_i \in H^0(U_i, TX)$ with *isolated zeroes* such that there exist non vanishing holomorphic functions (g_{ij}) on the intersections $U_i \cap U_j$ satisfying

$$v_i = g_{ij} v_j. \tag{1}$$

The locus defined by the vanishing of the local vector fields (v_i) is called the *singular locus* of $\mathcal{F}$ and denoted $\mathrm{Sing}(\mathcal{F})$.

The cocycle (g_{ij}) defines a line bundle $T^*\mathcal{F}$ on X, its dual is denoted $T\mathcal{F}$. Relation (1) means that the family (v_i) defines a section of $T^*\mathcal{F} \otimes TX$ and hence a sheaf map $T\mathcal{F} \to TX$. Two data $((U_i), (v_i))$, $((U'_j), (v'_j))$ are said to define the same foliation if the images of the associated sheaf maps are the same.

The line bundle $T\mathcal{F}$ is called the *tangent bundle* of the foliation and its dual $T^*\mathcal{F}$ is the *cotangent bundle* of $\mathcal{F}$. As defined, the line bundle $T\mathcal{F}$ is not canonically attached to $\mathcal{F}$, but its isomorphism class in the *Picard group* $\mathrm{Pic}(X)$ of X is.

One may also consider foliations on normal singular complex surfaces. They are defined by the datum of a foliation on the complement of the singular locus of the surface.

2.2. Rational vector fields and 1-forms. — If X is smooth projective, $T\mathcal{F}$ possesses a non trivial rational section and $\mathcal{F}$ can be given by a rational vector field $\mathcal{X}$, hence in $\mathrm{Pic}(X)$ we have

$$T\mathcal{F} = \mathcal{O}_X(\mathrm{div}(\mathcal{X})),$$

where $\mathrm{div}(\mathcal{X})$ denotes the divisor of zeroes and poles of $\mathcal{X}$. On a suitable (Zariski) open covering the local vector fields v_i are defined by setting $v_i = h\mathcal{X}_{|U_i}$, for a well chosen rational function h on U_i. This is how we associate a foliation to a simple derivation: we have a *preferred projective compactification of* $\mathbb{C}^2$, namely $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty, (x, y) \mapsto (x : y : 1)$, and a polynomial vector field on $\mathbb{C}^2$ extends to a rational vector field on $\mathbb{P}^2$.

One can also define a foliation by local holomorphic 1-forms with isolated zeroes (ω_i) that vanish on the local vector fields (v_i). If X is projective, such a family (ω_i) is obtained by locally eliminating poles and codimension 1 zeroes of a non trivial rational 1-form. Hence, on a smooth projective surface, a foliation may be defined by either a non trivial rational 1-form or a non trivial rational vector field.

2.3. Curves and foliations. — A curve C is termed *invariant by* $\mathcal{F}$ or $\mathcal{F}$-*invariant* if it is tangent to the local vector fields defining $\mathcal{F}$. When a compact curve $C \subset X$ is *not* $\mathcal{F}$-invariant we have the very useful *formula*

$$T^*\mathcal{F} \cdot C = \mathrm{tang}(\mathcal{F}, C) - C \cdot C,$$

where $\mathrm{tang}(\mathcal{F}, C)$ is the sum of orders of tangency between $\mathcal{F}$ and C, *cf.* [4, Prop. 2.2].