DIMER MODELS AND RANDOM TILINGS

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DIMER MODELS AND RANDOM TILINGS

Abstract. — *Les états de la Recherche* is a recurrent event organized by the French Mathematical Society. In 2009, a session was devoted to dimer models and random tilings. This volume regroups notes of some lectures given during this event, giving two different points of view on the topic. One is focused on the Kasteleyn approach; the other uses techniques of orthogonal polynomials, with analogies with random matrix theory.

Résumé. — *Les états de la Recherche* sont un événement récurrent organisé par la société mathématique de France. En 2009, une session était consacrée aux modèles de dimères et aux pavages aléatoires. Ce volume regroupe des notes de cours qui ont été donnés pendant cette session. Ils offrent deux points de vue différents sur le sujet. Le premier se concentre sur l'approche de Kasteleyn. Le deuxième utilise des techniques de polynômes orthogonaux, en lien avec la théorie des matrices aléatoires.

Société Mathématique de France

ÉTATS DE LA RECHERCHE

modèles de dimères et pavages aléatoires

Comité scientifique :
B. Duplantier (CEA Saclay)
R. Kenyon (Brown University)
Y. Le Jan (Université Paris-Sud)
W. Werner (Université Paris-Sud)

Organisateurs : C. Boutillier (UPMC), N. Enriquez (Université Paris X)

Béatrice de Tilière
(Université de Neuchâtel, Suisse)

Dimers and random surfaces: combinatorics and local statistics

Patrik Ferrari
(Universität Bonn, Allemagne)

Dimers and orthogonal polynomials: connections with random matrices

Nikolai Reshetikhin
(UC Berkeley, USA)

Dimers on surface graphs and random plane partitions

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http://ipht.cea.fr/statcomb2009/dimers/

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RÉSUMÉS DES ARTICLES

Le modèle de dimères consiste en l'étude de couplages parfaits aléatoires d'un graphe planaire, représentant la répartition de molécules di-atomiques à la surface d'un cristal. Ce modèle appartient au domaine plus large de la mécanique statistique. Le but de ces notes de cours est de donner un aperçu des résultats suivants : expression explicite pour la fonction de partition due à Kasteleyn, Temperley et Fisher ; interprétation en tant que fonction de hauteur des configurations de dimères d'un graphe biparti due à Thurston ; description complète du modèle de dimères sur un graphe infini, bi-périodique et biparti, due à Kenyon, Okounkov et Sheffield.

Dans ces notes, nous présentons quelques connexions entre les matrices aléatoires, le processus d'exclusion simple et les pavages aléatoires. D'abord, nous considérons l'ensemble gaussien unitaire des matrices aléatoires. Nous analysons la structure mathématique de leurs valeurs propres, ainsi que celles des mineurs principaux. Nous introduisons ensuite un système de particules en interaction en dimension 1 (le TASEP) dans lequel une structure mathématique similaire est exhibée. Enfin, nous étendons le système de particules à un modèle en dimension $2+1$. Ce modèle a une marginale donnée par le TASEP et sa projection à un instant fixé est une mesure sur des pavages aléatoires. Un cas spécial en temps discret donne le modèle bien connu du diamant aztèque.

ABSTRACTS

The dimer model in statistical mechanics
Béatrice de Tilière 1

The dimer model is the study of random perfect matchings of a planar graph, representing the adsorption of diatomic molecules on the surface of a crystal. It belongs to the field of statistical mechanics. The goal for these lectures is to give an overview of: the foundational results of Kasteleyn, Temperley and Fisher, proving an explicit formula for the partition funtion; Thurston's height function interpretation of dimer configurations of bipartite graphs; the paper "Dimers and Amoebae" of Kenyon, Okounkov and Sheffield, giving a full description of the dimer model on infinite, bi-periodic, bipartite graphs.

Dimers and orthogonal polynomials: connections with random matrices
Patrik L. Ferrari 47

In these lecture notes we present some connections between random matrices, the asymmetric exclusion process, random tilings. First we consider the Gaussian Unitary Ensemble of random matrices. We analyze the mathematical structure of their eigenvalues as well as the eigenvalues of the principal minors. Then we introduce a one-dimensional interacting particle system (the TASEP) where a similar mathematical structure arises. Finally we extend the particle system to a $2+1$ dimensional model. This model has a marginal given by the TASEP and the fixed time projection is a random tiling measure. A special case in discrete time gives the well-known Aztec diamond.

INTRODUCTION

The State of Research sessions (*États de la Recherche*) is a recurrent event of the French Mathematical Society, where each session is dedicated to a specific topic, with lectures aimed at specialists but also students and researchers wanting to discover a new field or wanting to know more about a specific topic.

In 2009, the topic of dimer models and random tilings has been chosen. More than 60 people from different communities: mathematicians, physicists and computer scientists attended this event, and followed mini-courses at a master level on various aspects of random tilings, coupled with research talks where original results were presented. This event was integrated in the schedule of the *Statcomb* semester(1) held at Institut Henri Poincaré, about statistical mechanics and combinatorics, organized by Mireille Bousquet-Mélou, Jérémie Bouttier, Gilles Schaeffer, Grégory Miermont.

The program of the event was chosen by the organizers and the scientific committee, composed of Bertrand Duplantier, Richard Kenyon, Yves Le Jan and Wendelin Werner.

Before giving a description of the content of the lectures and a summary of the talks, let us start with some definitions and some background about random tilings and dimer models.

1. Random tilings and dimer models

A *tile* is a compact set of the plane. A tiling of a region of the plane is a covering of that region with tiles without holes or overlaps: the union of the tiles is the whole region, and the intersection of their interiors is empty.

A particular subclass of planar tilings is constructed as follows: the region to tile is the union of bounded faces of a planar graph, and the set of possible tiles is the union of two adjacent (closed) faces of this graph. Those tiling models are called *dimer models*, by analogy between the tiles and diatomic molecules, the role of the atoms being played by the faces of the graph. Two particular examples of dimer models are domino tilings, where tiles are 2×1 rectangles (two neighboring faces of

(1) `http://ipht.cea.fr/statcomb2009/`.

the square lattice) and lozenge tilings, where rhombic tiles are two neighboring faces of the triangular lattice glued together.

Typically, tiling problems are NP-difficult: time needed to a computer to check if a certain placement of tiles is indeed a tiling of a region grows polynomially with the size of the region. However, time needed for known algorithms to produce a solution in the first place grows too rapidly with the size of the region to be used in practice on very large regions.

On the contrary, dimer models are *easy*, from this point of view: it takes a polynomial time to determine whether or not a given region admits a tiling with dimers. Even more, it is possible to count the exact number of configurations in polynomial time. This surprising fact is a consequence of a result obtained simultaneously by Kasteleyn [**8**], and Fisher and Temperley [**18**], stating that this number of configurations is expressed as the determinant of some matrix, describing how the small faces of the region are connected one to another.

The dimer models were the subject of many theorems proven in the beginning of the century. In the investigation of some scaling limits, limiting laws were obtained which already appeared in the theory of random matrices, a subject of probability which also grew a lot in the recent years. Incidentally, it was discovered that dimer models have deep connections with other branches of mathematics, outside of combinatorics and probability: discrete differential geometry and discrete complex analysis, topology of surfaces, representation theory, real algebraic geometry...

The goal of this session was to provide an overview of the various methods which have been successfully used to derive precise results about these models and explain some of those connections with other fields.

2. Content of the lectures

2.1. Mini-courses. – Three minicourses were given by Béatrice de Tilière, Patrik Ferrari and Nikolai Reshetikhin. The lecture notes of the first two courses are reproduced in this volume.

After having given some definitions of the model, Béatrice de Tilière gave in the first part of her course a proof of Kasteleyn's theorem [**8**], which gives an expression of the partition function of the dimer model on a bipartite planar graph as the determinant of a signed adjacency matrix of the graph, the *Kasteleyn matrix*. She also explained, following Kenyon [**9**] how to compute correlations for k edges as a $k \times k$ determinant of a submatrix of the inverse of the Kasteleyn matrix, thanks to a Jacobi identity. She then gave an interpretation of a random bipartite dimer model as a random interface, by associating to each dimer configuration a *height function*. In the second part, she presented the results obtained by Kenyon, Okounkov and Sheffield [**12**] about the classification of ergodic Gibbs measures for dimer models on biperiodic bipartite planar graphs. The understanding of the phase diagram for the model, and the universality results about the decay of correlations in the three phases (*solid*, *liquid*, *gaseous*) relies on the fact that the *spectral curve* of the dimer model, the zero locus of a polynomial constructed from the combinatorics of the graph and the weights, is a Harnack curve(2) and as such has some positivity property. See her lecture notes for the details of the construction.

Patrick Ferrari provided some background about random matrix theory and the Gaussian Unitary Ensemble (GUE), with an emphasis on the GUE minor process and its interlacing structure. Then, he introduced the *totally asymmetric exclusion process* (TASEP), a system of particles sharing the same structure as the GUE minor process. For both models, explicit computations can be performed using the theory of orthogonal polynomials. Finally, he defined an extension of the TASEP as a particle system in $2+1$-dimensional space-time. This model admits as marginal the original TASEP and a discrete analog of the Dyson Brownian motion, the dynamic version of the GUE, where Gaussian static random variables are replaced by Brownian motions. The configuration of this particle system at a fixed time is also directly related to a dimer model. The positions of the particles encode a tiling by dominos of a region of the plane, called the Aztec diamond. When time evolves, the dynamics of the particles reflects the *domino shuffling*, a random algorithm to transform a tiling of an Aztec diamond into one of a slightly larger Aztec diamond, by moving dominos around and filling the holes with new dominos. This extended model gives an explanation why the GUE, TASEP and domino tilings of the Aztec tilings have a similar structure, since they are marginals of the same process.

The course of Nicolai Reshetikhin was focused on two of the connections with other branches of mathematics listed above. In his first lecture, he defined the Schur processes.

A *partition* λ is a nonincreasing sequence of positive integers $(\lambda_i)_{i\geq 1}$, with a finite size $|\lambda| = \sum_i \lambda_i < +\infty$. Two partitions are said to be *interlaced* if $\lambda \prec \mu$, i.e.: $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots$, or $\mu \prec \lambda$. Schur processes are probability measures on sequences of interlaced partitions.

A *plane partition* π is the natural two-dimensional analogue: it is a two-dimensional array $(\pi_{i,j})_{i,j\geq 1}$ with nonincreasing rows and columns, and a finite size $|\pi| = \sum_{i,j} \pi_{i,j} < +\infty$. A plane partition can be represented graphically as piles of cubes, each pile corresponding to an entry of the plane partition. This notion can be generalized to a *skew plane partition*, where the indices i, j of the two-dimensional array π do not run freely among all integers, but are of the form: $i \geq 1, j \geq \lambda_i$, with λ a partition, and the same monotonicity constraint on rows and columns of π.

Due to this monotonicity, the surface of the heap of cubes can be projected properly onto the plane orthogonal to the direction $(1,1,1)$ to get a tiling with rhombi.

(2) Conversely, every Harnack curve is the spectral curve of a dimer model, which gives an explicit paramaterization of the set of Harnack curves, see **[11]**.