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SPACE-TIME PARAPRODUCTS FOR PARACONTROLLED CALCULUS, 3D-PAM AND MULTIPLICATIVE BURGERS EQUATIONS

BY Ismael BAILLEUL, Frédéric BERNICOT AND Dorothee FREY

ABSTRACT. – We sharpen in this work the tools of paracontrolled calculus in order to provide a complete analysis of the parabolic Anderson model equation and Burgers system with multiplicative noise, in a 3-dimensional Riemannian setting, in either bounded or unbounded domains. With that aim in mind, we introduce a pair of intertwined space-time paraproducts on parabolic Hölder spaces, with good continuity, that happens to be pivotal and provides one of the building blocks of higher order paracontrolled calculus.

RÉSUMÉ. – Nous enrichissons dans ce travail les outils du calcul paracontrôlé afin de fournir une analyse complète de l'équation du modèle parabolique d'Anderson et du système Burgers avec un bruit multiplicatif, dans un cadre riemannien de dimension 3, dans des domaines bornés ou non. Dans ce but, nous introduisons une paire de paraproduits espace-temps agissant sur les espaces de Hölder paraboliques, qui se révèle cruciale et fournit l'un des éléments constitutifs du calcul paracontrôlé d'ordre supérieur.

1. Introduction

It is probably understated to say that the work [19] of Hairer has opened a new era in the study of stochastic singular parabolic partial differential equations. It provides a setting where one can make sense of a product of a distribution with parabolic non-positive Hölder regularity index, say a, with a function with non-negative regularity index, say b, even in the case where $a + b$ is non-positive, and where one can make sense of, and solve, a large class of parabolic stochastic singular partial differential equations by fixed point methods. The parabolic Anderson model equation (PAM)

$$(\partial_t + L)u = u\zeta, \tag{1.1}$$

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studied in Section 5 in a 3-dimensional unbounded background, is an example of such an equation, as it makes sense in that setting to work with a distribution ζ of Hölder exponent $\alpha - 2$, for some $\alpha < \frac{1}{2}$, while one expects the solution u to the equation to be of parabolic Hölder regularity α, making the product $u\zeta$ ill-defined since $\alpha + (\alpha - 2) \leq 0$.

The way out of this quandary found by Hairer has its roots in Lyons' theory of rough paths, which already faced the same problem. Lyons' theory addresses the question of making sense of, and solving, controlled differential equations

$$dz_t = V_i(z_t)\, dX_t^i \tag{1.2}$$

in $\mathbb{R}^d$ say, driven by an $\mathbb{R}^\ell$-valued $\frac{1}{p}$-Hölder control $X = \big(X^1, \dots, X^\ell\big)$, with $p \geq 2$, and where V_i are sufficiently regular vector fields on $\mathbb{R}^d$. Typical realizations of a Brownian path are $\frac{1}{p}$-Hölder continuous, with $p > 2$, for instance. One expects a solution path to equation (1.2) to be $\frac{1}{p}$-Hölder continuous as well, in which case the product $V_i(z_t)\, dX_t^i$, or the integral $\int_0^t V_i(z_s)\, dX_s^i$, cannot be given an intrinsic meaning since $\frac{1}{p} + \big(\frac{1}{p} - 1\big) \leq 0$. Lyons' deep insight was to realize that one can make sense of, and solve, equation (1.2) if one assumes one is given an enriched version of the driving signal X that formally consists of X together with its non-existing iterated integrals. The theory of regularity structures rests on the same philosophy, and the idea that the enriched noise should be used to give a local description of the unknown u, in the same way as polynomials are used to define and describe locally C^k functions.

At the very same time that Hairer built his theory, Gubinelli, Imkeller and Perkowski proposed in [17] another implementation of that philosophy building on a different notion of local description of a distribution, using paraproducts on the torus. The machinery of paracontrolled distributions introduced in [17] rests on a first order Taylor expansion of a distribution that happened to be sufficient to deal with the stochastic parabolic Anderson equation (1.1) on the 2-dimensional torus, the stochastic additive Burgers equation in one space dimension [17], the Φ_3^4 equation on the 3-dimensional torus [12, 28] and the stochastic Navier-Stokes equation with additive noise [26, 27]. The KPZ equation can also be dealt with using this setting [18]. Following Bony's approach [9], the paraproduct used in [17] is defined in terms of Fourier analysis and does not allow for the treatment of equations outside the flat background of the torus or the Euclidean space, if one is ready to work with weighted functional spaces. The geometric restriction on the background was greatly relaxed in our previous work [3] by building paraproducts from the heat semigroup associated with the operator L in the semilinear equation. A theory of paracontrolled distributions can then be considered in doubling metric measure spaces where one has small time Gaussian estimates on the heat kernel and its 'gradient'—see [3]. This setting already offers situations where the theory of regularity structures is not known to be working. The stochastic parabolic Anderson model equation in a 2-dimensional doubling manifold was considered in [3] as an example. The first order 'Taylor expansion' approach of paracontrolled calculus seems however to restrict a priori its range of application, compared to the theory of regularity structures, and it seems clear that a kind of higher order paracontrolled calculus is needed to extend its scope. We tackle in the present work the first difficulty that shows off in this program, which is related to the crucial use of commutator estimates between the heat operator and a paraproduct, which is one of the three workhorses of the paracontrolled calculus method, together with Schauder estimates and another continuity result on some commutator. The development of a high order paracontrolled calculus is the object of another work [4].

Working in unbounded spaces with weighted functional spaces requires a careful treatment which was not done so far. We shall illustrate the use of our machinery on two examples: The parabolic Anderson model (PAM) equation (1.1) in a possibly unbounded 3-dimensional Riemannian manifold, and Burgers equation with multiplicative noise in the 3-dimensional closed Riemannian manifold. Hairer and Labbé have very recently studied the (PAM) equation in $\mathbb{R}^3$ from the point of view of regularity structures [21]—see also the work [22] of Hairer and Pardoux. They had to introduce some weights ϖ to get a control on the growth of quantities of interest at spatial infinity. A non-trivial part of their work consists into tracking the time-behavior of their estimates, with respect to the time, which requires the use of time-dependent weights. For the same reason, we also need to use weighted spaces and working with the weights of [22, 21] happens to be convenient. Our treatment is however substantially easier, as we do not need to travel backwards in time such as required in the analysis of the reconstruction operator in the theory of regularity structures. As a matter of fact, our results on the (PAM) equation give an alternative approach, and provide a non-trivial extension, of the results of [21] to a non-flat setting, with a possibly wider range of operators L than can be treated presently in the theory of regularity structures. As for Burgers equation with multiplicative noise, it provides a description of the random evolution of a velocity field on the 3-dimensional torus, subject to a random rough multiplicative forcing, and whose dynamics reads

$$(\partial_t + L)u + (u \cdot \nabla)u = \mathrm{M}_\zeta u, \tag{1.3}$$

where ζ is a 3-dimensional white noise with independent coordinates, and

$$\mathrm{M}_\zeta u := \big(\zeta^1 u^1, \zeta^2 u^2, \zeta^3 u^3\big),$$

for the velocity field $u = \big(u^1, u^2, u^3\big) : M^3 \to \mathbb{R}^3$. With zero noise ζ, this 3-dimensional Burgers system plays a very important role in the theory of PDEs coming from fluid mechanics, and later from condensed matter physics and statistical physics. It has been proposed by Burgers in the 30's as a simplified model of dynamics for Navier-Stokes equations. A change of variables, called after Cole and Hopf, can be used to reduce the deterministic quasilinear parabolic equation to the heat equation, thus allowing the derivation of exact solutions in closed form. Despite this fact, the study of Burgers system is still very fashionable as a benchmark model that can be used to understand the basic features of the interaction between nonlinearity and dissipation. Motivated by the will to turn Burgers equation into a model for turbulence, stochastic variants have been the topic of numerous recent works [8, 23, 24, 19, 17, 18], where a random forcing term is added in the equation, mainly in one space dimension, with an additive space-time white noise—that is with a space-time white noise instead of $\mathrm{M}_\zeta u$ with ζ space white noise. The Cole-Hopf transformation can formally be used again, and turns a solution to the 1-dimensional stochastic Burgers equation with additive space-time noise to the heat equation with multiplicative space-time noise, with a very singular noise, such as detailed in [18]. A similar change of variable trick can be used for the study of the above multidimensional stochastic Burgers