ISSN 0012-9593

quatrième série - tome 52 fascicule 1 janvier-février 2019

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Christophe GARBAN & Hugo VANNEUVILLE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

Édition et abonnements / Publication and subscriptions

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : `abonnements@smf.emath.fr`

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 52, 2019, p. 1 à 57

EXCEPTIONAL TIMES FOR PERCOLATION UNDER EXCLUSION DYNAMICS

BY Christophe GARBAN AND Hugo VANNEUVILLE

Abstract. – We analyze in this paper a conservative analog of the celebrated model of *dynamical percolation* introduced by Häggström, Peres and Steif in [10]. It is simply defined as follows: start with an initial percolation configuration $\omega(t = 0)$. Let this configuration evolve in time according to a simple exclusion process with symmetric kernel $K(x, y)$. We start with a general investigation (following [10]) of this dynamical process $t \mapsto \omega_K(t)$ which we call K*-exclusion dynamical percolation*. We then proceed with a detailed analysis of the planar case at the critical point (both for the triangular grid and the square lattice $\mathbb{Z}^2$) where we consider the power-law kernels K^α

$$K^\alpha(x, y) \propto \frac{1}{\|x - y\|_2^{2+\alpha}}.$$

We prove that if $\alpha > 0$ is chosen small enough, there exist *exceptional times* t for which an infinite cluster appears in $\omega_{K^\alpha}(t)$. (On the triangular grid, we prove that this holds for all $\alpha < \alpha_0 = \frac{217}{816}$.) The existence of such exceptional times for standard i.i.d. dynamical percolation (where sites evolve according to independent Poisson point processes) goes back to the work by Schramm-Steif in [25]. In order to handle such a K-exclusion dynamics, we push further the spectral analysis of *exclusion noise sensitivity* which has been initiated in [3]. (The latter paper can be viewed as a conservative analog of the seminal paper by Benjamini-Kalai-Schramm [1] on i.i.d. noise sensitivity.) The case of a nearest-neighbor simple exclusion process, corresponding to the limiting case $\alpha = +\infty$, is left widely open.

Résumé. – Cet article porte sur une version conservative du modèle de la *percolation dynamique* introduit par Häggström, Peres et Steif dans [10]. Le modèle se définit simplement de la façon suivante : on tire une configuration de percolation initiale $\omega(t = 0)$. Puis, on fait évoluer cette configuration selon un processus d'exclusion simple de noyau symétrique $K(x, y)$. On commence par une étude générale (en suivant [10]) du processus $t \mapsto \omega_K(t)$ que l'on appelle *percolation dynamique sous K-exclusion*. Nous analysons ensuite de façon détaillée le cas bi-dimensionnel au point critique (à la fois pour le réseau triangulaire et pour le réseau $\mathbb{Z}^2$) pour des noyaux en loi de puissance K^α

$$K^\alpha(x, y) \propto \frac{1}{\|x - y\|_2^{2+\alpha}}.$$

The first author has been supported by the ANR grant Liouville 15-CE40-0013 and the ERC grant LiKo 676999.

 doi:10.24033/asens.2383

Nous montrons que si l'exposant $\alpha > 0$ est suffisamment petit, il existe des *temps exceptionnels t* pour lesquels une composante connexe infinie se forme dans $\omega_{K^\alpha}(t)$. (Pour la percolation par site sur réseau triangulaire, on montre que cela se produit pour tout $\alpha < \alpha_0 = \frac{217}{816}$). L'existence de tels temps exceptionnels pour la percolation dynamique standard i.i.d. (où les sites évoluent selon des processus de Poisson indépendants) remonte au travail de Schramm-Steif [25]. Afin de contrôler la dynamique ci-dessus du type K-exclusion, on approfondit l'analyse spectrale de la *sensibilité au bruit sous exclusion* initiée dans le travail [3]. (Travail qui est en quelque sorte l'analogue conservatif du papier précurseur par Benjamini-Kalai-Schramm [1] sur la sensibilité au bruit i.i.d.). Le cas du processus d'exclusion simple au plus proche voisin, correspondant au cas limite $\alpha = +\infty$, reste entièrement ouvert.

1. Introduction

1.1. Dynamical percolation

We consider bond percolation on an infinite, countable, connected, locally finite graph $G = (V, E)$. We write $\mathbb{P}_p$ for the probability measure of (bond) *percolation of parameter p* on G i.e., the probability measure on $\Omega = \{-1, 1\}^E$ obtained by declaring each edge *open* with probability p and *closed* with probability $1 - p$, independently of the others (1 means open and -1 means closed). More formally, $\mathbb{P}_p$ is the product measure $(p\delta_1 + (1 - p)\delta_{-1})^{\otimes E}$ on Ω equipped with the product σ-algebra. An element $\omega \in \Omega$ is called a *percolation configuration*. Moreover, a connected component of the graph obtained by keeping only the open edges is called a *cluster*. It is a simple consequence of Kolmogorov's 0-1 law that, for each p, $\mathbb{P}_p$ [$\exists$ an infinite cluster] $\in \{0, 1\}$. Moreover, it is well known (see for instance [9] or [2]) that there exists a *critical point* $p_c = p_c(G) \in [0, 1]$ such that:

$$\forall p \in [0, p_c), \mathbb{P}_p\,[\exists \text{ an infinite cluster}] = 0,$$
$$\forall p \in (p_c, 1], \mathbb{P}_p\,[\exists \text{ an infinite cluster}] = 1.$$

The most studied model is bond percolation on the Euclidean lattice $\mathbb{Z}^d$, $d \geq 2$. For this model, it is known that $p_c = p_c(d) \in (0, 1)$. In other words, there exists a phase transition. Moreover, it is a celebrated theorem by Kesten [15] that $p_c(2) = 1/2$ and it is conjectured that, for any $d \geq 2$, $\mathbb{P}_{p_c}$ [$\exists$ an infinite cluster] $= 0$. This last property has been proved for $d = 2$ ([13]) and $d \geq 11$ (see [12, 4]).

In [10], Häggström, Peres and Steif define and study the model of *dynamical* (bond) *percolation* (this model was invented independently by Benjamini). Dynamical percolation of parameter $p \in [0, 1]$ is defined very easily as follows: we sample a percolation configuration $\omega(0)$ according to some initial law and we then let evolve each edge independently of each other according to Poisson point processes: at rate one, the states of edges are resampled using $p\delta_1 + (1 - p)\delta_{-1}$. We obtain this way a càdlàg Markov process $(\omega(t))_{t\geq 0}$ on the space Ω (seen as the compact metric product space) with $\mathbb{P}_p$ as (unique) invariant probability measure. The main question is whether, if $\omega(0) \sim \mathbb{P}_p$ (1), there exist *exceptional times* for which the percolation configuration is very atypical. Exceptional times are defined as follows: if $\mathbb{P}_p$ [$\exists$ an infinite cluster] $= 0$, then an exceptional time is a time for which there

(1) Where $X \sim P$ means that P is the distribution of the random variable X.

is an infinite cluster. On the other hand, if $\mathbb{P}_p$ [$\exists$ an infinite cluster] $= 1$, then it is a time for which there is no infinite cluster.

From now on, we assume that $\omega(0) \sim \mathbb{P}_p$. Since $\mathbb{P}_p$ is an invariant measure, then (by Fubini) a.s. Leb-a.e. there is no exceptional time (where Leb is the Lebesgue measure on $\mathbb{R}_+$). This does not imply that a.s. there does not exist any exceptional time. However, this is the case away from the critical point: the authors of [10] have proved that, for any graph G, if $p \neq p_c$ then a.s. there is no exceptional time (see their Proposition 1.1).

The case $p = p_c$ is in general much more difficult. First, let us note that, for bond percolation on the Euclidean lattice $\mathbb{Z}^d$, this is for now interesting only for $d = 2$ and $d \geq 11$ since these are the only dimensions for which we know what happens at criticality. For $d \geq 11$, thanks to a result proved in [12] for $d \geq 19$ (and extended very recently to $d \geq 11$ in [4]), the authors of [10] have proved that, even at criticality, a.s. there is no exceptional time (see their Theorem 1.3). However, for $d = 2$, the following is proved in [5] (Theorem 1.4):

for dynamical bond percolation on $\mathbb{Z}^2$, a.s. there are exceptional times if $p = p_c = 1/2$.

Such a result had been proved earlier in [25] for the model of *site percolation on the triangular lattice*. Let $\mathbb{T}$ denote the (planar) triangular lattice and let $\mathbb{P}_p$ denote the probability measure of site percolation on $\mathbb{T}$ (this is the analogous model where the sites—i.e., the vertices of $\mathbb{T}$—are open or closed; in this context a cluster is a connected component of the graph obtained by keeping only the open sites). Kesten's work also implies that $p_c = 1/2$ for this model. Of course, one can define dynamical site percolation on $\mathbb{T}$ in the same way as for dynamical bond percolation i.e., by associating exponential clocks to the sites of $\mathbb{T}$. Much more is known for site percolation on $\mathbb{T}$ than for bond percolation on $\mathbb{Z}^2$. Indeed conformal invariance (as the mesh goes to zero) has been proved by Smirnov in [26], and the exact value of several critical exponents (see Subsection 2.1) has been derived in [18, 27] using the *Schramm Loewner Evolution (SLE)* processes introduced by Schramm. Using the knowledge of these critical exponents, the following is proved in [25] (Theorem 1.3):

For dynamical site percolation on $\mathbb{T}$, a.s. there are exceptional times if $p = p_c = 1/2$.

Finally, let us mention that in [5] it is shown that, for critical site percolation on $\mathbb{T}$, the Hausdorff dimension of the set of exceptional times is a.s. $31/36$. For other results, see for instance [11] where the authors show that typical exceptional times are intimately related to the so-called Incipient Infinite Cluster introduced by Kesten.

In both [25] and [5], the main methods are related to the theory of *Fourier decomposition of Boolean functions*. In the present paper, we will also rely extensively on such tools, see Subsections 2.3 and 2.4.

1.2. Percolation under exclusion dynamics

We study in this paper the same question of existence of exceptional times but with a different underlying dynamical process: we let the configuration evolve according to a *symmetric exclusion process*. Percolation evolving according to an exclusion process has already been studied by Broman, the first author and Steif in [3] where the authors introduce and study the notion of *exclusion sensitivity*. (We will say more about this notion in Section 2, see also [22, 21].) To define and study a symmetric exclusion process (which is a Feller Markov