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CONVEX PROJECTIVE GENERALIZED DEHN FILLING

BY SUHYOUNG CHOI, GYE-SEON LEE AND LUDOVIC MARQUIS

ABSTRACT. – For $d = 4, 5, 6$, we exhibit the first examples of complete finite volume hyperbolic d-manifolds M with cusps such that infinitely many d-orbifolds M_m obtained from M by generalized Dehn filling admit properly convex real projective structures. The orbifold fundamental groups of M_m are Gromov-hyperbolic relative to a collection of subgroups virtually isomorphic to $\mathbb{Z}^{d-2}$, hence the images of the developing maps of the projective structures on M_m are new examples of divisible properly convex domains of the projective d-space which are not strictly convex, in contrast to the previous examples of Benoist.

RÉSUMÉ. – En dimension $d = 4, 5, 6$, on construit les premiers exemples de variétés hyperboliques M complètes de volume fini de dimension d avec des pointes telles qu'une infinité d'orbifolds M_m obtenues par remplissage de Dehn généralisé sur M admettent des structures projectives convexes. Les groupes fondamentaux au sens des orbifolds des M_m sont Gromov-hyperboliques relativement à une collection de sous-groupes virtuellement isomorphes à $\mathbb{Z}^{d-2}$. Ainsi les images des applications développantes associées aux structures projectives sur les orbifolds M_m fournissent des nouveaux exemples de convexes divisibles non-strictement convexes de dimension d qui contrastent avec les exemples précédents de Benoist.

1. Introduction

1.1. Motivation

Hyperbolic Dehn filling theorem proven by Thurston [49] is a fundamental theorem of hyperbolic 3-manifold theory. It states that if the interior of a compact 3-manifold M with toral boundary admits a complete hyperbolic structure of finite volume, then except for finitely many Dehn fillings on each boundary component, all other Dehn fillings of M admit hyperbolic structures.

Within the realm of hyperbolic geometry, this phenomenon happens only for *three*-manifolds even though there is a notion of topological Dehn filling for every compact d-manifold M with boundary homeomorphic to the $(d-1)$-dimensional torus $\mathbb{T}^{d-1}$. Let

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$\mathbb{D}^n$ be the closed n-ball. Since the compact d-manifold $\mathbb{D}^2 \times \mathbb{T}^{d-2}$ has also a torus boundary component, if we glue M and $\mathbb{D}^2 \times \mathbb{T}^{d-2}$ together along their boundaries, then we obtain closed d-manifolds, called *Dehn fillings of M*. In a similar way we can deal with compact d-manifolds with multiple torus boundary components. Now even if we assume that the interior of M admits a finite volume hyperbolic structure, no Dehn fillings of M admit a hyperbolic structure when $d \geqslant 4$ (see Theorem 1.7).

Although Dehn fillings in dimension bigger than 3 admit no hyperbolic structure, they can admit a geometric structure which is larger than a hyperbolic structure. A main aim of this article is to show that a real projective structure can be a good candidate for this purpose. In a similar spirit, Anderson [2] and Bamler [6] proved that many aspects of Dehn filling theory for hyperbolic 3-manifolds can be generalized to Einstein d-manifolds in every dimension $d \geqslant 3$. More precisely, if the interior of a compact d-manifold M with toral boundary admits a finite volume hyperbolic structure, then except for finitely many Dehn fillings on each boundary component, all other Dehn fillings of M admit Einstein metrics.

Let $\mathbb{S}^d$ be the d-dimensional projective sphere (1) and note that the group $\mathrm{SL}^{\pm}_{d+1}(\mathbb{R})$ of linear automorphisms of $\mathbb{R}^{d+1}$ with determinant ± 1 is the group of projective automorphisms of $\mathbb{S}^d$. We say that a d-dimensional manifold N *admits a properly convex real projective structure* if N is homeomorphic to Ω/Γ with Ω a properly convex subset of $\mathbb{S}^d$ and Γ a discrete subgroup of $\mathrm{SL}^{\pm}_{d+1}(\mathbb{R})$ acting properly discontinuously on Ω. Now we can ask the following question:

QUESTION 1.1. – *Is there a compact manifold M of dimension $d \geqslant 4$ with toral boundary such that the interior of M admits a finite volume hyperbolic structure, and except for finitely many Dehn fillings on each boundary component, all other Dehn fillings of M admit properly convex real projective structures?*

REMARK 1.2. – We can generalize Question 1.1 only requiring the interior of M to admit a properly convex real projective structure of finite volume.

1.2. Evidence

In this paper, we give an evidence towards a positive answer to Question 1.1. It is difficult for us to find such a manifold directly, and hence we begin with Coxeter orbifold, also called reflection orbifold or reflectofold (see [49, 1, 20] for the definition of orbifold). The definition of Coxeter orbifold is somewhat more complicated than the definition of manifold, however it turns out that convex projective Coxeter orbifolds are easier to build than convex projective manifolds. The first examples towards Question 1.1 are hyperbolic Coxeter d-orbifolds for $d = 4, 5, 6$.

The projective model of the hyperbolic d-space $\mathbb{H}^d$ is a round open ball in the projective d-sphere $\mathbb{S}^d$. Consider a (convex) polytope P in an affine chart of $\mathbb{S}^d$ such that every facet (2) of P has a non-empty intersection with $\mathbb{H}^d$ and all dihedral angles of P are submultiples of π, i.e., each dihedral angle is $\frac{\pi}{m}$ with an integer $m \geqslant 2$ or $m = \infty$. We call such a polytope a *hyperbolic Coxeter polytope*. The group Γ generated by the reflections about the facets

(1) The *projective d-sphere* $\mathbb{S}^d$ is the space of half-lines of $\mathbb{R}^{d+1}$.
(2) A *facet* of a polytope is a face of codimension 1.

of P is a discrete subgroup of the group $\mathrm{Isom}(\mathbb{H}^d)$ of isometries of $\mathbb{H}^d$, which is isomorphic to $\mathrm{O}^+_{d,1}(\mathbb{R})$, and the quotient $\mathbb{H}^d/\Gamma$ is a hyperbolic Coxeter d-orbifold. Hyperbolic Coxeter orbifolds are useful objects that we can concretely construct. For example, the first example of a closed orientable hyperbolic 3-manifold is an eight-fold cover of a right-angled hyperbolic Coxeter 3-orbifold with 14 faces, which was constructed by Löbell [39] in 1931.

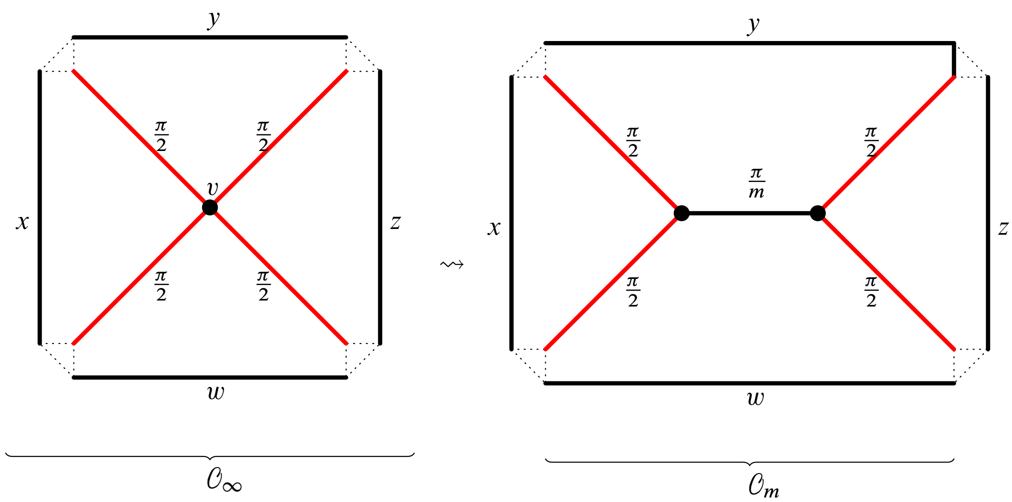

FIGURE 1. Hyperbolic Dehn fillings in dimension 3.

By Andreev's theorems [3, 4], there is a version of Dehn filling theorem for hyperbolic Coxeter 3-orbifolds (see Chapter 7 of Vinberg and Shvartsman [56] and Proposition 2 of Kolpakov [36]): Let $\mathcal{O}_\infty$ be a compact Coxeter 3-orbifold with boundary $\partial\mathcal{O}_\infty$ that is a closed Coxeter 2-orbifold admitting a Euclidean structure. An *m-generalized Dehn filling* $\mathcal{O}_m$ of $\mathcal{O}_\infty$, or simply an *m-Dehn filling*, is a closed Coxeter 3-orbifold $\mathcal{O}_m$ such that $\mathcal{O}_\infty$ is orbifold diffeomorphic [(3)] to the complement of an open neighborhood of an edge r of order [(4)] m of $\mathcal{O}_m$ (see Figure 1 and Definition 3.1). A corollary of Andreev's theorems says that if the interior of $\mathcal{O}_\infty$ admits a hyperbolic structure of finite volume, then there exists a natural number N such that for each $m > N$, the 3-orbifold $\mathcal{O}_m$ admits a hyperbolic structure. Note that the existence of a hyperbolic m-Dehn filling implies that the boundary $\partial\mathcal{O}_\infty$ of $\mathcal{O}_\infty$ must be a quadrilateral with four right angles as one can see on Figure 1 (see Proposition 9.1 for a higher dimensional and projective version of this statement).

Now we can state the main theorem of the paper: let $\mathcal{O}_\infty$ be a compact Coxeter d-orbifold with boundary $\partial\mathcal{O}_\infty$ that is a closed Coxeter $(d-1)$-orbifold admitting a Euclidean structure. An *m-Dehn filling* $\mathcal{O}_m$ of $\mathcal{O}_\infty$ is a Coxeter d-orbifold such that $\mathcal{O}_\infty$ is orbifold diffeomorphic to the complement of an open neighborhood of a ridge [(5)] r of $\mathcal{O}_m$, and each interior point

(3) See Davis [26] or Wiemeler [57].
(4) An edge r of $\mathcal{O}_m$ is said to be *of order m* if each interior point of r has a neighborhood modeled on $(\mathbb{R}^2/D_m)\times\mathbb{R}$, where D_m is the dihedral group of order $2m$ generated by reflections in two lines meeting at angle $\frac{\pi}{m}$.
(5) A *ridge* of a polytope is a face of codimension 2.