ASTÉRISQUE 280

COHOMOLOGY OF SIEGEL VARIETIES

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Société Mathématique de France 2002
Publié avec le concours du Centre National de la Recherche Scientifique

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2000 *Mathematics Subject Classification.* — 11F46, 11G15, 14F30, 14K22, 17B50, 17B56, 20G30.

***Key words and phrases.* —** Siegel modular forms, p-adic cohomology, Galois representations, BGG complexes, (co)homology of nilpotent groups.

COHOMOLOGY OF SIEGEL VARIETIES

Abdellah Mokrane, Patrick Polo, Jacques Tilouine

***Abstract.* —** This volume deals with the question of torsion in the cohomology of Shimura varieties, the coefficient system being $\mathbb{Z}_p$ or, more generally, a certain local system V_λ of flat $\mathbb{Z}_p$-modules. Its goal is to show, for Siegel varieties, that the localization of this cohomology at a non-Eisenstein maximal ideal $\mathfrak{m}$ of the Hecke algebra $\mathbb{T}$ has no p-torsion ($p = \mathrm{char}(\mathbb{T}/\mathfrak{m})$), for p greater than an explicit bound $c(\lambda)$ depending only on the highest weight λ of the coefficient system. This localization, moreover, kills the boundary cohomology.

Two arithmetic applications are presented: one concerns Hida families of Hecke eigensystems and the other is a step towards the existence of certain Taylor-Wiles systems for symplectic groups.

An ingredient in the proof is a version over $\mathbb{Z}_p$ of Bernstein-Gelfand-Gelfand complexes and of Kostant's theorem computing the $\mathfrak{n}$-homology of the Weyl module V_λ, for p greater than the above bound $c(\lambda)$ (which implies that λ belongs to the closure of the fundamental p-alcove).

***Résumé* (Cohomologie des variétés de Siegel). —** Cette monographie traite de la question de la torsion dans la cohomologie des variétés de Shimura, à coefficients dans $\mathbb{Z}_p$ ou, plus généralement, dans un certain système local V_λ de $\mathbb{Z}_p$-modules plats. Son objet est d'établir, pour les variétés de Siegel, que la localisation de cette cohomologie en un idéal maximal de type non-Eisenstein $\mathfrak{m}$ de l'algèbre de Hecke $\mathbb{T}$ n'a pas de p-torsion ($p = \mathrm{char}(\mathbb{T}/\mathfrak{m})$), pour p plus grand qu'une certaine borne explicite $c(\lambda)$ qui ne dépend que du plus haut poids λ du système de coefficients. En outre, cette localisation tue la cohomologie du bord.

On donne deux applications arithmétiques de ce résultat. L'une concerne les familles de Hida de systèmes de valeurs propres de Hecke, l'autre constitue une étape importante dans la construction de certains systèmes de Taylor-Wiles pour les groupes symplectiques.

Un ingrédient de la preuve est une version sur $\mathbb{Z}_p$ de complexes de Bernstein-Gelfand-Gelfand et d'un théorème de Kostant, calculant la $\mathfrak{n}$-homologie du module de Weyl V_λ, pour p plus grand que la borne ci-dessus (ce qui implique que λ appartient à l'adhérence de la p-alcôve fondamentale).

CONTENTS

INTRODUCTION

The first paper of this volume deals with the question of torsion in the cohomology of Siegel varieties S_U with coefficients in a local system V_λ of finite flat $\mathbb{Z}_p$-modules. Its goal is to show that its localization at a non-Eisenstein maximal ideal $\mathfrak{m}$ of the Hecke algebra is torsion-free for p large with respect to the highest weight λ of the coefficient system V_λ. At the same time, as could be expected, besides getting rid of the torsion, the localization has the effect of killing the boundary cohomology (and its torsion), so that we show that

$$H_c^\bullet(S_U, V_\lambda)_\mathfrak{m} = IH^\bullet(S_U, V_\lambda)_\mathfrak{m} = H_!^\bullet(S_U, V_\lambda)_\mathfrak{m} = H^\bullet(S_U, V_\lambda)_\mathfrak{m}$$

and these cohomology modules are concentrated in middle degree d. This question of absence of torsion is important in the construction of p-ordinary families of cuspidal Hecke eigensystems, and in the verification of the first main condition for having a Taylor-Wiles system. These applications are given at the end of the paper. The main assuption is that the p-adic Galois representation associated to a cuspidal cohomological representation does exist (it is known only in genus $\leqslant 2$) and that those corresponding to the maximal ideal $\mathfrak{m}$ have large residual image (it can be verified on examples for $g = 2$). Faltings introduced around 1980 the dual Bernstein-Gelfand-Gelfand complex as a tool for determining the Hodge decomposition of the complex cohomology of locally symmetric varieties. The rational version of this tool appeared in Faltings-Chai book, and they incidentally mention that a p-adic integral version as well exists, but only for so-called p-small weights λ. We developed this idea, and it became our main tool for determining the Fontaine-Laffaille constituents of the modulo p de Rham cohomology of these Siegel varieties. This, allied with Falting's mod. p étale-de Rham comparison theorem together with a Galois-theoretic argument allows us to show the vanishing of the various modulo p cohomologies of S_U localized at $\mathfrak{m}$ in degree $q < d$. For this, we needed a rather detailed study of the Bernstein-Gelfand-Gelfand complex over $\mathbb{Z}_p$ (in p-small weight) and a $\mathbb{Z}_p$-integral version of Kostant theorem decomposing the cohomology of the unipotent radical of a parabolic as a sum of irreducible modules over the Levi quotient. These results are presented in great generality in the second paper, which provides also a useful assortment of results on $\mathbb{Z}_p$-representations of a reductive group in p-small weights.

Astérisque
280, 2002, p. 1–95

COHOMOLOGY OF SIEGEL VARIETIES WITH p-ADIC INTEGRAL COEFFICIENTS AND APPLICATIONS

by

Abdellah Mokrane & Jacques Tilouine

Abstract. — Under the assumption that Galois representations associated to Siegel modular forms exist (it is known only for genus at most 2), we study the cohomology with p-adic integral coefficients of Siegel varieties, when localized at a non-Eisenstein maximal ideal of the Hecke algebra, provided the prime p is large with respect to the weight of the coefficient system. We show that it is torsion-free, concentrated in degree d, and that it coincides with the interior cohomology and with the intersection cohomology. The proof uses p-adic Hodge theory and the dual BGG complex modulo p in order to compute the "Hodge-Tate weights" for the mod. p cohomology. We apply this result to the construction of Hida p-adic families for symplectic groups and to the first step in the construction of a Taylor-Wiles system for these groups.

Résumé **(Cohomologie des variétés de Siegel à coefficients entiers p-adiques et applications)**
Supposant connue l'existence des représentations galoisiennes associées aux formes modulaires de Siegel (elle ne l'est qu'en genre $\leqslant 2$ pour le moment), on étudie la cohomologie des variétés de Siegel à coefficients entiers p-adiques localisée en un idéal maximal non-Eisenstein de l'algèbre de Hecke, lorsque p est grand par rapport au poids du système de coefficients. Plus précisément, on montre qu'elle est sans torsion, concentrée en degré médian, et qu'elle coïncide avec la cohomologie d'intersection et avec la cohomologie intérieure. On utilise pour cela la théorie de Hodge p-adique et le complexe BGG dual modulo p qui calcule « les poids de Hodge-Tate » de la réduction modulo p de cette cohomologie. On applique ce résultat à la construction de familles de Hida p-ordinaires pour les groupes symplectiques et à l'ébauche de la construction d'un système de Taylor-Wiles pour ces groupes.

1. Introduction

1.1. Let G be a connected reductive group over $\mathbb{Q}$. Diamond [**16**] and Fujiwara [**29**] (independently) have axiomatized the Taylor-Wiles method which allows to study some local components $\mathbf{T}_{\mathfrak{m}}$ of a Hecke algebra $\mathbf{T}$ for G of suitable (minimal) level; when it applies, this method shows at the same time that $\mathbf{T}_{\mathfrak{m}}$ is complete intersection and that some cohomology module, viewed as a $\mathbf{T}$-module, is locally free at $\mathfrak{m}$. It

2000 ***Mathematics Subject Classification.*** — 11F46, 11G15, 14K22, 14F30.
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has been successfully applied to $\mathrm{GL}(2)_{/\mathbb{Q}}$ [**73**], to some quaternionic Hilbert modular cases [**29**], and to some inner forms of unitary groups [**38**]. If one tries to treat other cases, one can let the Hecke algebra act faithfully on the middle degree Betti cohomology of an associated Shimura variety; then, one of the problems to overcome is the possible presence of torsion in the cohomology modules with p-adic integral coefficients. For $G = \mathrm{GSp}(2g)$ ($g \geqslant 1$), we want to explain in this paper why this torsion is not supported by maximal ideals of $\mathbf{T}$ which are "non-Eisenstein" and ordinary (see below for precise definitions), provided the residual characteristic p is prime to the level and greater than a natural bound. A drawback of our method is that it necessitates to assume that the existence and some local properties of the Galois representations associated to cohomological cuspidal representations on G are established. For the moment, they are proven for $g \leqslant 2$ (see below). In his recent preprint [**43**], Hida explains for the same symplectic groups G how by considering only coherent cohomology, one can let the Hecke algebra act faithfully too on cohomology modules whose torsion-freeness is built-in (without assuming any conjecture). However for some applications (like the relation, for some groups G, between special values of adjoint L-functions, congruence numbers, and cardinality of adjoint Selmer groups), the use of the Betti cohomology seems indispensable.

1.2. Let $G = \mathrm{GSp}(2g)$ be the group of symplectic similitudes given by the matrix $J = \left(\begin{smallmatrix} 0 & s \\ -s & 0 \end{smallmatrix}\right)$, whose entries are $g \times g$-matrices, and s is antidiagonal, with non-zero coefficients equal to 1; the standard Borel B, resp. torus T, in G consists in upper triangular matrices, resp. diagonal matrices in G. For any dominant weight λ for (G, B, T), we write $\widehat{\lambda}$ for its dual (that is, the dominant weight associated to the Weyl representation dual of that of λ). Let ρ be the half-sum of the positive roots. Recall that λ is given by a $(g+1)$-uple $(a_g, \ldots, a_1; c) \in \mathbb{Z}^{g+1}$ with $c \equiv a_1 + \cdots + a_g \bmod 2$, that $\widehat{\lambda} = (a_g, \ldots, a_1; -c)$ and $\rho = (g, \ldots, 1; 0)$ (see section 3.1 below). Throughout this paper, the following integer will be of great importance:

$$\boldsymbol{w} = |\lambda + \rho| = |\lambda| + d = \sum_{i=1}^{g}(a_i + i) = d + \sum_{i=1}^{g} a_i$$

where $d = g(g+1)/2$. It can be viewed as a cohomological weight as follows.

Let $\mathbb{A} = \mathbb{A}_f \times \mathbb{Q}_\infty$ be the ring of rational adèles; let G_f resp. G_∞ be the group of $\mathbb{A}_f$-points resp. $\mathbb{Q}_\infty$-points of G. Let U be a "good" open compact subgroup of $G(\mathbb{A}_f)$ (see Introd. of Sect. 2); let S resp. S_U be the Shimura variety of infinite level, resp. of level U associated to G; then $d = \dim S_U$ is the middle degree of the Betti cohomology of S_U. Let $V_\lambda(\mathbb{C})$ be the coefficient system over S resp. S_U with highest weight λ. See Sect. 2.1 for precise definitions.

Let $\pi = \pi_f \otimes \pi_\infty$ be a cuspidal automorphic representation of $G(\mathbb{A})$ which occurs in $H^d(S_U, V_\lambda(\mathbb{C}))$. This means that