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THE PRO-ÉTALE TOPOLOGY FOR SCHEMES

by

Bhargav Bhatt & Peter Scholze

To Gérard Laumon, with respect and admiration

Abstract. — We give a new definition of the derived category of constructible $\overline{\mathbf{Q}}_\ell$-sheaves on a scheme, which is as simple as the geometric intuition behind them. Moreover, we define a refined fundamental group of schemes, which is large enough to see all lisse $\overline{\mathbf{Q}}_\ell$-sheaves, even on non-normal schemes. To accomplish these tasks, we define and study the pro-étale topology, which is a Grothendieck topology on schemes that is closely related to the étale topology, and yet better suited for infinite constructions typically encountered in ℓ-adic cohomology. An essential foundational result is that this site is locally contractible in a well-defined sense.

***Résumé* (La topologie pro-étale sur les schémas).** — On donne une nouvelle définition de la catégorie dérivée des $\overline{\mathbf{Q}}_\ell$-faisceaux constructibles sur un schéma, qui est aussi simple que l'intuition géométrique sous-jacente. De plus, on définit sur les schémas un groupe fondamental raffiné qui est assez grand pour voir tous les $\overline{\mathbf{Q}}_\ell$-faisceaux lisses, même sur les schémas qui ne sont pas normaux. Pour obtenir cela, on définit et étudie la topologie pro-étale, qui est une topologie de Grothendieck sur les schémas étroitement liée à la topologie étale mais mieux adaptée aux constructions infinies typiques de la cohomologie ℓ-adique. Un résultat de base essentiel est que ce site est localement contractile en un sens bien défini.

1. Introduction

Let X be a variety over an algebraically closed field k. The étale cohomology groups $H^i(X_{\text{ét}}, \overline{\mathbf{Q}}_\ell)$, where ℓ is a prime different from the characteristic of k, are of fundamental importance in algebraic geometry. Unfortunately, the standard definition of these groups is somewhat indirect. Indeed, contrary to what the notation suggests, these groups are not obtained as the cohomology of a sheaf $\overline{\mathbf{Q}}_\ell$ on the étale site $X_{\text{ét}}$.

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The étale site gives the correct answer only with torsion coefficients, so the correct definition is

$$H^i(X_{\text{ét}}, \overline{\mathbf{Q}}_\ell) := \big(\varprojlim_n H^i(X_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z})\big) \otimes_{\mathbf{Z}_\ell} \overline{\mathbf{Q}}_\ell\ .$$

In this simple situation, this technical point is often unproblematic [(1)]. However, even here, it takes effort to construct a natural commutative differential graded $\overline{\mathbf{Q}}_\ell$-algebra giving rise to these cohomology groups. This so-called $\overline{\mathbf{Q}}_\ell$-homotopy type was constructed by Deligne in [**Del80**], using certain subtle integral aspects of homotopy theory due independently to Miller [**Mil78**] and Grothendieck.

For more sophisticated applications, however, it is important to work in a relative setup (*i.e.*, study constructible sheaves), and keep track of the objects in the derived category, instead of merely the cohomology groups. In other words, one wants a well-behaved derived category $D^b_c(X, \overline{\mathbf{Q}}_\ell)$ of constructible $\overline{\mathbf{Q}}_\ell$-sheaves. Deligne, [**Del80**], and in greater generality Ekedahl, [**Eke90**], showed that it is possible to define such a category along the lines of the definition of $H^i(X_{\text{ét}}, \overline{\mathbf{Q}}_\ell)$. Essentially, one replaces $H^i(X_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z})$ with the derived category $D^b_c(X, \mathbf{Z}/\ell^n\mathbf{Z})$ of constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-sheaves, and then performs all operations on the level of categories [(2)]:

$$D^b_c(X, \overline{\mathbf{Q}}_\ell) := \big(\varprojlim_n D^b_c(X, \mathbf{Z}/\ell^n\mathbf{Z})\big) \otimes_{\mathbf{Z}_\ell} \overline{\mathbf{Q}}_\ell\ .$$

Needless to say, this presentation is oversimplified, and veils substantial technical difficulties.

Nonetheless, in daily life, one pretends (without getting into much trouble) that $D^b_c(X, \overline{\mathbf{Q}}_\ell)$ is simply the full subcategory of some hypothetical derived category $D(X, \overline{\mathbf{Q}}_\ell)$ of all $\overline{\mathbf{Q}}_\ell$-sheaves spanned by those bounded complexes whose cohomology sheaves are locally constant along a stratification. Our goal in this paper to justify this intuition, by showing that the following definitions recover the classical notions. To state them, we need the pro-étale site $X_{\text{proét}}$, which is introduced below. For any topological space T, one has a 'constant' sheaf on $X_{\text{proét}}$ associated with T; in particular, there is a sheaf of (abstract) rings $\overline{\mathbf{Q}}_\ell$ on $X_{\text{proét}}$ associated with the topological ring $\overline{\mathbf{Q}}_\ell$.

Definition 1.1. — *Let X be a scheme whose underlying topological space is noetherian.*

1. *A sheaf L of $\overline{\mathbf{Q}}_\ell$-modules on $X_{\text{proét}}$ is* lisse *if it is locally free of finite rank.*
2. *A sheaf C of $\overline{\mathbf{Q}}_\ell$-modules on $X_{\text{proét}}$ is* constructible *if there is a finite stratification $\{X_i \to X\}$ into locally closed subsets $X_i \subset X$ such that $C|_{X_i}$ is lisse.*

1. It becomes a problem as soon as one relaxes the assumptions on k, though. For example, even for $k = \mathbf{Q}$, this definition is not correct: there is no Hochschild-Serre spectral sequence linking these naively defined cohomology groups of X with those of $X_{\overline{k}}$. One must account for the higher derived functors of inverse limits to get a theory linked to the geometry of $X_{\overline{k}}$, see [**Jan88**].

2. In fact, Ekedahl only defines the derived category of constructible $\mathbf{Z}_\ell$-sheaves, not performing the final $\otimes_{\mathbf{Z}_\ell}\overline{\mathbf{Q}}_\ell$-step.

3. *An object $K \in D(X_{\text{proét}}, \overline{\mathbf{Q}}_\ell)$ is* constructible *if it is bounded, and all cohomology sheaves are constructible. Let $D_c^b(X, \overline{\mathbf{Q}}_\ell) \subset D(X_{\text{proét}}, \overline{\mathbf{Q}}_\ell)$ be the corresponding full triangulated subcategory.*

The formalism of the six functors is easily described in this setup. In particular, in the setup above, *with the naive interpretation of the right-hand side*, one has

$$H^i(X_{\text{ét}}, \overline{\mathbf{Q}}_\ell) = H^i(X_{\text{proét}}, \overline{\mathbf{Q}}_\ell) \ ;$$

for general X, one recovers Jannsen's continuous étale cohomology, [**Jan88**]. Similarly, the complex $R\Gamma(X_{\text{proét}}, \overline{\mathbf{Q}}_\ell)$ is obtained by literally applying the derived functor $R\Gamma(X_{\text{proét}}, -)$ to a sheaf of $\mathbf{Q}$-algebras, and hence naturally has the structure of a commutative differential graded algebra by general nonsense (see [**Ols11**, §2] for example); this gives a direct construction of the $\overline{\mathbf{Q}}_\ell$-homotopy type in complete generality.

A version of the pro-étale site was defined in [**Sch13**] in the context of adic spaces. The definition given there was somewhat artificial, mostly because non-noetherian adic spaces are not in general well-behaved. This is not a concern in the world of schemes, so one can give a very simple and natural definition of $X_{\text{proét}}$. Until further notice, X is allowed to be an arbitrary scheme.

Definition 1.2

1. *A map $f : Y \to X$ of schemes is* weakly étale *if f is flat and $\Delta_f : Y \to Y \times_X Y$ is flat.*
2. *The* pro-étale site *$X_{\text{proét}}$ is the site of weakly étale X-schemes, with covers given by fpqc covers.*

Any map between weakly étale X-schemes is itself weakly étale, and the resulting topos has good categorical properties, like coherence (if X is qcqs) and (hence) existence of enough points. For this definition to be useful, however, we need to control the class of weakly étale morphisms. In this regard, we prove the following theorem.

Theorem 1.3. — *Let $f : A \to B$ be a map of rings.*

1. *f is étale if and only if f is weakly étale and finitely presented.*
2. *If f is ind-étale,* i.e., *B is a filtered colimit of étale A-algebras, then f is weakly étale.*
3. *If f is weakly étale, then there exists a faithfully flat ind-étale $g : B \to C$ such that $g \circ f$ is ind-étale.*

In other words, for a ring A, the sites defined by weakly étale A-algebras and by ind-étale A-algebras are equivalent, which justifies the name pro-étale site for the site $X_{\text{proét}}$ defined above. We prefer using weakly étale morphisms to define $X_{\text{proét}}$ as the property of being weakly étale is clearly étale local on the source and target, while that of being ind-étale is not even Zariski local on the target.