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OUTLINE OF THE PROOF OF THE GEOMETRIC LANGLANDS CONJECTURE FOR GL_2

by

Dennis Gaitsgory

To Gérard Laumon

Abstract. — We outline a proof of the categorical geometric Langlands conjecture for GL_2, as formulated in [**AG**], modulo a number of more tractable statements that we call Quasi-Theorems.

***Résumé* (Les grandes lignes de la démonstration de la correspondance de Langlands géométrique pour GL_2).** — On donne les grandes lignes d'une démonstration de la conjecture de Langlands géométrique catégorique pour GL_2, telle que formulée dans [**AG**], modulo un certain nombre d'énoncés davantage à portée que nous appelons des quasi-théorèmes.

Introduction

0.1. The goal of this paper. — The goal of this paper is to describe work-in-progress by D. Arinkin, V. Drinfeld and the author [(1)] towards the proof of the (categorical) geometric Langlands conjecture.

The contents of the paper can be summarized as follows: we reduce the geometric Langlands conjecture to a combination of two sets of statements.

The first set is what we call "quasi-theorems". These are plausible (and tractable) statements that involve Langlands duality, but either for proper Levi subgroups, or of local nature, or both. Hopefully, these quasi-theorems will soon turn into actual theorems.

The second set are two conjectures (namely, Conjectures 8.2.9 and 10.2.8), both of which are theorems for GL_n. However, these conjectures *do not* involve Langlands

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1. The responsibility for any deficiency or undesired outcome of this paper lies with the author of this paper.

duality: Conjecture 8.2.9 only involves the geometric side of the correspondence, and Conjecture 10.2.8 only the spectral side.

0.2. Strategy of the proof. — In this subsection we will outline the general scheme of the argument. We will be working over an algebraically closed field k of characteristic 0. Let X be a smooth and complete curve over k, and G a reductive group. We let $\check{G}$ denote the Langlands dual group, also viewed as an algebraic group over k.

0.2.1. Formulation of the conjecture. — The categorical geometric Langlands conjecture is supposed to compare two triangulated (or rather DG categories). One is the "geometric" (or "automorphic") side that has to do with D-modules on the stack Bun_G of G-bundles on X. The other is the "spectral" (or "Galois") side that has to do with quasi-coherent sheaves on the stack $\mathrm{LocSys}_{\check{G}}$ on $\check{G}$-local systems on X.

In our formulation of the conjecture, the geometric side is taken "as is". *I.e.*, we consider the DG category $\text{D-mod}(\mathrm{Bun}_G)$ of D-modules on Bun_G. We refer the reader to [**DrGa2**] for the definition of this category and a discussion of its general properties (*e.g.*, this category is *compactly generated* for non-tautological reasons).

A naive guess for the spectral side is the DG category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$. However, this guess turns out to be slightly wrong, whenever G is not a torus. A quick way to see that it is wrong is *via* the compatibility of the conjectural geometric Langlands equivalence with the functor of Eisenstein series, see Property Ei stated in Sect. 6.4.8. Namely, if P is a parabolic of G with Levi quotient M, we have the Eisenstein series functors

$$\begin{gathered}\mathrm{Eis}_P : \text{D-mod}(\mathrm{Bun}_M) \longrightarrow \text{D-mod}(\mathrm{Bun}_G) \quad \text{and} \\ \mathrm{Eis}_{\check{P},\mathrm{spec}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}) \longrightarrow \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}),\end{gathered}$$

that are supposed to match up under the geometric Langlands equivalence (up to a twist by some line bundles). However, this cannot be the case because the functor Eis_P preserves compactness (see [**DrGa3**]), whereas $\mathrm{Eis}_{\check{P},\mathrm{spec}}$ does not.

Our "fix" for the spectral side is designed to make the above problem with Eisenstein series go away in a minimal way (see Proposition 6.4.7). We observe that the non-preservation of compactness by the functor $\mathrm{Eis}_{\check{P},\mathrm{spec}}$ has to do with the fact that the stack $\mathrm{LocSys}_{\check{G}}$ is not smooth. Namely, it expresses itself in that some coherent complexes on $\mathrm{LocSys}_{\check{G}}$ are non-perfect.

Our modified version for the spectral side is the category that we denote

$$\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}),$$

see Sect. 3.3.2. It is a certain enlargement of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$, whose definition uses the fact that $\mathrm{LocSys}_{\check{G}}$ is a derived locally complete intersection, and the theory of singular support of coherent sheaves for such stacks developed in [**AG**].

0.2.2. Idea of the proof. — The idea of the comparison between the categories $\text{D-mod}(\text{Bun}_G)$ and $\text{IndCoh}_{\text{Nilp}^{\text{glob}}_{\check{G}}}(\text{LocSys}_{\check{G}})$ pursued in this paper is the following: we embed each side into a more tractable category and compare the essential images.

For the geometric side, the more tractable category in question is the category that we denote $\text{Whit}^{\text{ext}}(G,G)$, and refer to it as the *extended Whittaker category*; the nature of this category is explained in Sect. 0.2.3 below. The functor

$$\text{D-mod}(\text{Bun}_G) \longrightarrow \text{Whit}^{\text{ext}}(G,G)$$

(which, according to Conjecture 8.2.9, is supposed to be fully faithful) is that of *extended Whittaker coefficient*, denoted $\text{coeff}^{\text{ext}}_{G,G}$.

For the spectral side, the more tractable category is denoted $\text{Glue}(\check{G})_{\text{spec}}$, and the functor

$$\text{IndCoh}_{\text{Nilp}^{\text{glob}}_{\check{G}}}(\text{LocSys}_{\check{G}}) \longrightarrow \text{Glue}(\check{G})_{\text{spec}}$$

is denoted by $\text{Glue}(\text{CT}^{\text{enh}}_{\text{spec}})$ (this functor is fully faithful by Theorem 9.3.8). The idea of the pair $(\text{Glue}(\check{G})_{\text{spec}}, \text{Glue}(\text{CT}^{\text{enh}}_{\text{spec}}))$ is explained in Sect. 0.2.4.

We then claim (see Quasi-Theorems 9.4.2 and 9.4.5) that there exists a canonically defined fully faithful functor

$$\mathbb{L}^{\text{Whit}^{\text{ext}}}_{G,G} : \text{Glue}(\check{G})_{\text{spec}} \longrightarrow \text{Whit}^{\text{ext}}(G,G).$$

Thus, we have the following diagram

$$\begin{array}{ccc} \text{Glue}(\check{G})_{\text{spec}} & \xrightarrow{\mathbb{L}^{\text{Whit}^{\text{ext}}}_{G,G}} & \text{Whit}^{\text{ext}}(G,G) \\ {\scriptstyle \text{Glue}(\text{CT}^{\text{enh}}_{\text{spec}})}\uparrow & & \uparrow{\scriptstyle \text{coeff}^{\text{ext}}_{G,G}} \\ \text{IndCoh}_{\text{Nilp}^{\text{glob}}_{\check{G}}}(\text{LocSys}_{\check{G}}) & & \text{D-mod}(\text{Bun}_G), \end{array} \tag{0.1}$$

with all the arrows being fully faithful.

Assume that the essential images of the functors

$$\mathbb{L}^{\text{Whit}^{\text{ext}}}_{G,G} \circ \text{Glue}(\text{CT}^{\text{enh}}_{\text{spec}}) \text{ and } \text{coeff}^{\text{ext}}_{G,G} \tag{0.2}$$

coincide. We then obtain that diagram (0.1) can be (uniquely) completed to a commutative diagram by means of a functor

$$\mathbb{L}_G : \text{IndCoh}_{\text{Nilp}^{\text{glob}}_{\check{G}}}(\text{LocSys}_{\check{G}}) \longrightarrow \text{D-mod}(\text{Bun}_G),$$

and, moreover, $\mathbb{L}_G$ is automatically an equivalence.

The required fact about the essential images of the functors (0.2) follows from Conjecture 10.2.8.