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ON THE DEFINITION OF THE GALOIS GROUPOID

by

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For José Manuel Aroca on the occasion of his 60th birthday

Abstract. — We sketch a proof of equivalence of two general differential Galois theories, Malgrange's theoy and ours, if the base field consists only of constants.

***Résumé* (Sur la définition du groupoïde de Galois).** — Nous esquissons la démonstration du fait que deux théories de Galois, la théorie de Malgrange et la nôtre, sont équivalentes dans le cas absolu, i.e. quand le corps de base consiste uniquement en des constantes.

1. Introduction

Today we have two general differential Galois theories [**4**] and [**3**]. While the first published in 1996 is a Galois theory of differential field extensions, the latter proposed in 2001 is a Galois theory of foliations on varieties. They look somehow different but specialists observed coincidence in examples. The aim of this note is to sketch in fact they are equivalent in the absolute case, by which we mean the case where the base field K of the differential field extension L/K consists of only constants. For the relative case or for a general differential field extension L/K, there may be a similar result but there are subtle questions. First of all we must have an adequate definition of the Galois groupoid for the extension L/K in terms of foliations in the spirit of [**3**].(1)

We show by analyzing a non-trivial interesting example, the equivalence. Given a differential field, it is an algebraic counter part of a dynamical system on a algebraic variety. If we observe this dynamical system closely by algebraic method, or if an algebraist observes the dynamical system, then we get as a natural object Galois groupoid of the dynamical system, or of the given differential field. This procedure of

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(1) Added on 25 August 2008, we can apply this method also to the reative case.

observation is done through the universal Taylor morphism and ties Malgrange's idea and ours.

2. Differential fields and dynamical systems

A differential field (L, δ) consists of a field L and a derivation $\delta : L \to L$. So we have $\delta(a+b)) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for every $a, b \in L$. Similarly we define a differential ring (R, δ). An element a of a differential field or a differential ring is called a constant if $\delta(a) = 0$. The set C_L or C_R of constants forms respectively a subfield or subring.

Now we consider a differential field that is finitely generated as an abstract field over the complex number field $\mathbb{C}$ in such a way that the complex number field $\mathbb{C}$ is a subfield of the field C_L of constants.

Remark 2.1. — *In the sequel, we work over the complex number field $\mathbb{C}$ so that the reader has a concrete image, we may replace, however, the complex number field $\mathbb{C}$ by any field of characteristic* 0.

We explain by examples that a differential field is an algebraic counter part of a differential dynamical system on an algebraic variety.

Example 2.1. Let us consider the differential field $(\mathbb{C}(x), d/dx)$, where x is a variable over $\mathbb{C}$ and hence $\mathbb{C}(x)$ is the rational function field of one variable. A geometric model of the differential field $(\mathbb{C}(x), d/dx)$ is a dynamical system $(\mathbb{A}^1, d/dx) =$ (Spec $\mathbb{C}[x]$, d/dx). In other words, the field of rational functions of the affine line $\mathbb{A}^1$ with derivation d/dx gives the differential field $(\mathbb{C}(x), d/dx)$.

Remark 2.2. — *Since for any non-empty Zariski open subset U of $\mathbb{A}^1$, $(U, d/dx)$ satisfies the condition required above, the general model of the differential field $(\mathbb{C}(x), d/dx)$ is $(\mathbb{A}^1-$ (a finite number of points), $d/dx)$. The model is determined up to birational equivalence.*

Example 2.2. Let x, y be two independent variables over $\mathbb{C}$ so that $\mathbb{C}[x, y]$ is a polynomial ring over $\mathbb{C}$. Let us consider the differential field

$$(\mathbb{C}(x, y), \partial/\partial x + y\partial/\partial y).$$

A model of this differential field is the (x, y)-plane $\mathbb{A}^2$ or Spec $\mathbb{C}[x, y]$ with vector field $\partial/\partial x + y\partial/\partial y$. A general flow on the affine plane $\mathbb{A}^2$ is given by $(t, c\exp t)$, $t \in \mathbb{C}$ for a fixed $c \in \mathbb{C}$. In this Example we may replace the affine plane $\mathbb{A}^2$ by any non-empty Zariski open set of $\mathbb{A}^2$.

Generally we can prove the following proposition.

Proposition 2.1. — *Let (L, δ) be a differential field such that the field L is of finite type over the complex number field $\mathbb{C}$ and $\mathbb{C}$ is a subfield of the field C_L of constants of (L, δ). Then there exists a smooth algebraic variety V over $\mathbb{C}$, with regular algebraic vector field X such that (V, X) is a model of the differential field (L, δ). In other* *words , the rational function field $\mathbb{C}(V)$ of V is isomorphic to the field L and the vector field X is identified with the derivation δ through this isomorphism.*

See Lemma (1.5), [**5**].

3. Groupoids

We need a seemingly abstract definition of groupoid but it is as concrete as vector space.

Definition 3.1. — *A groupoid is a small category G in which all morphisms are isomorphisms. An object of G is called a vertex and a morphism in G is called an element of G.*

The groupoid was introduced by Brandt in 1926. In 1950's Ehresmann used groupoids in theory of foliations. In 1960's Grothendieck studied quotients by groupoids in algebraic geometry. Here are examples of groupoids to have an image of groupoids.

Example 3.1. A group G is a groupoid. We define a category $\mathscr{C}$ that is a groupoid. The object of the category $\mathscr{C}$ is one point P, i.e. $ob\,\mathscr{C} = \{P\}$. We set

$$\mathrm{Hom}\ (P,\, P) = G$$

and compose two morphisms of $\mathrm{Hom}\ (P,\, P) = G$ according as the group law of G.

Example 3.2. Equivalence relation $\sim$ on a set X. The set $ob\,G$ of the objects of the groupoid G is the set X. For $x,\, y \in ob\,G$, we define

$$\mathrm{Hom}\ (x,\, y) = \begin{cases} 1 \text{ morphism}, & \text{if } x \sim y, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Since every element x is equivalent to itself, we have the identity Id_x. Since equivalence relation is reflexive, every morphism is an isomorphism. Since equivalence relation is transitive, we can compose two morphisms. So the above definition yields us a groupoid.

Example 3.3. Group operation (G, X) of a group G on a set X is a groupoid. The set $ob\,C$ of the groupoid C is the set X. For $x,\, y \in X = ob\,C$, we set $\mathrm{Hom}\ (x,\, y) = \{g \in G | gx = y\}$. If $g \in\ \mathrm{Hom}\ (x,\, y)$ and $h \in\ \mathrm{Hom}\ (y,\, z)$, then $gx = y$ and $hy = z$ by definition so that $z = hy = h(gx) = (hg)x$ and consequently $hg \in\ \mathrm{Hom}\ (x, z)$. So we can compose two morphisms. If $gx = y$, then $hy = x$, h being g^{-1} so that every morphism is an isomorphism.

Example 3.4. Poincaré groupoid. Let X be a topological space. Let $ob\,G$ of the category be the set X. A path from a point $x \in X$ to another point $y \in X$ is a continuous map $\varphi : [0, 1] \to X$ from the interval $[0, 1]$ to the topological space X such that $\varphi(0) = x$ and $\varphi(1) = y$. We set in the category G,

$$\text{Hom } (x, y) := \text{ the set of paths from } x \text{ to } y \text{ modulo homotopy equivalence.}$$

Then it is well-know that the category G is a groupoid, which is called a Poincaré groupoid.

Now let G be a groupoid . We set

$$Y := \{\text{morphisms in the category } G\}$$

and

$$X := ob\, G.$$

Let $\varphi \in Y$ so that $\varphi \in$ Hom (A, B) for some $A, B \in ob\, G$. Let us denote the source A of φ by $s(\varphi)$ and the target B of φ by $t(\varphi)$. So we get two maps $s : Y \to X$ and $t : Y \to X$. Let $(Y, t) \times (Y, s)$ be the fiber product of $t : Y \to X$ and $s : Y \to X$ so that

$$(Y, t) \times (Y, s) = \{(\varphi, \psi) \in Y \times Y | s(\varphi) = t(\psi)\}.$$

The composition of morphisms defines a map

$$\Phi : (Y, t) \times (Y, s) \to Y, \qquad (\varphi, \psi) \mapsto \psi \circ \varphi.$$

The associativity of the composition is described by a commutative diagram that we do not make precise. See [**2**]. The existence of the identity map Id_A for every $A \in ob\, G$ as well as the property called symmetry that every morphism is an isomorphism is also characterized in terms of maps and commutative diagrams.

Here is a summary of the above observation. Groupoid is described by two sets Y and X, two maps $s : Y \to X$ and $t : Y \to X$ and the composition maps

$$\Phi : (Y, t) \times (Y, s) \to Y, \qquad (\varphi, \psi) \mapsto \psi \circ \varphi.$$

that satisfy certain commutative diagrams and so on.

This allows us to generalize the notion of groupoid in a category in which fiber product exists. This is exactly by the same way as we define an algebraic group G requiring that, first of all, G is an algebraic variety, the composition law $G \times G \to G$ is a morphism of algebraic varieties and so on.

Definition 3.2. — *Let C be a category in which fiber product exists. A groupoid in the category C consists of two objects Y, $X \in ob\, C$, two morphisms $s : Y \to X$ and $t : Y \to X$ and a morphism*

$$\Phi : (Y, t) \times (Y, s) \to Y$$

etc, satisfying the above conditions (cf. Grothendieck [**2**]*)*

Example 3.5. Let C be the category of algebraic varieties defined over a field k and let (G, V) be an operation of an algebraic group on an algebraic variety V defined over k. We have two morphisms p, h from $G \times V$ to V, namely the second projection

p and the group operation $h(g, v) = gv$. Then $Y = G \times X$, $X = V$ $s = p$ and $t = h$ is a groupoid in the category C. Compare to Example 3.3.

We need a tool, an algebraic D-groupoid that generalizes Example 3.5.

4. Lie groupoids and D-groupoids

For a complex manifold V, we can attach its invertible jets $J^*(V \times V)$ that is a groupoid over $V \times V$ in the category of analytic spaces. We recall the definition for $V = \mathbb{C}$. The jet space $J(\mathbb{C} \times \mathbb{C})$ is an infinite dimensional analytic space $\mathbb{C} \times \mathbb{C}^{\mathbb{N}}$ with coordinate system $(x, , y_0, y_1, y_2, \dots)$, We have two morphisms $s : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C}$ and $t : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C}$ given by

$$s((x, , y_0, y_1, y_2, \dots)) = x \qquad \textit{and} \qquad t((x, , y_0, y_1, y_2, \dots)) = y_0.$$

So we have a morphism $(s, t) : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C} \times \mathbb{C}$ that makes $J(\mathbb{C} \times \mathbb{C})$ an infinite dimensional affine space over $\mathbb{C} \times \mathbb{C}$. The invertible jet space $J^*(\mathbb{C} \times \mathbb{C})$ is , by definition, the Zarisiki open set of $J(\mathbb{C} \times \mathbb{C})$. Namely,

$$J^*(\mathbb{C} \times \mathbb{C}) := \{(x, y_0, y_1, y_2, \dots) \in J(\mathbb{C} \times \mathbb{C}) | \, y_1 \neq 0\}.$$

We simply denote $J^*(\mathbb{C} \times \mathbb{C})$ by J^* and we write the restrictions of the morphisms s, t to the Zariski open set J^* by the same letters. Now we explain J^* with two morphisms $s : J^* \to \mathbb{C}$ and $t : J^* \to \mathbb{C}$ is a groupoid. To this end we must define the composite morphism $\Phi : (J^*, t) \times (J^*, s) \to J^*$. Let

$$\varphi = (x, y_0, y_1, \dots), \qquad \psi = (u, v_0, v_1, \dots),$$

be points of J^* such that $y_0 = t(\varphi) = s(\psi) = u$, i.e. (φ, ψ) is a point of $(J^*, t) \times (J^*, s)$. Then we set

$$\Phi(\psi, \varphi) := (x, v_0, y_1 v_1, y_2 v_1 + y_1^2 v_2, \dots). \tag{1}$$

The n-th component of $\Phi(\psi, \varphi)$ is given by the following rule. Imagine formally that φ were a function of x taking the value y_0 at x, or $\varphi(x) = y_0$, with $\varphi'(x) = y_1, \varphi''(x) = y_2 \dots$. Similarly consider as if ψ were a function of u with $\psi(u) = v_0, \psi'(u) = v_1, \psi''(u) = v_2, \dots$. Then $\Phi(\psi, \varphi)$ is the composite function $\psi \circ \varphi$, which is a function of x, so that its n-th component is the value of $d^n \psi \circ \varphi / dx^n$ at x. For example ,

$$d(\psi \circ \varphi)/dx = \psi_u \varphi_x = y_1 v_1, d^2(\psi \circ \varphi)/dx^2 = \varphi_{xx} \psi_u + \varphi_x^2 \psi_{uu} = y_2 v_1 + y_1^2 u_2, \dots$$

One can check this composition law is associative and the inverse of

$$\varphi = (x, y_0, y_1, \dots)$$

is given by the inverse function $x(y_0)$ and its derivatives $d^n x(y_0)/dy_0^n$ for $n \in \mathbb{N}$, namely by

$$(y_0, x, 1/y_1, -y_2/y_1^3, \dots).$$

We can very naturally extend this construction over a complex manifold of any dimension.