Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

NLS ON DOMAINS

Ramona Anton

Tome 136
Fascicule 1

2008

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 27-65

Bull. Soc. math. France
136 (1), 2008, p. 27–65

STRICHARTZ INEQUALITIES FOR LIPSCHITZ METRICS ON MANIFOLDS AND NONLINEAR SCHRÖDINGER EQUATION ON DOMAINS

by Ramona Anton

Abstract. — We prove wellposedness of the Cauchy problem for the nonlinear Schrödinger equation for any defocusing power nonlinearity on a domain of the plane with Dirichlet boundary conditions. The main argument is based on a generalized Strichartz inequality on manifolds with Lipschitz metric.

Résumé (*Inégalités de Strichartz pour des métriques lipschitziennes et équation de Schrödinger non-linéaire sur des domaines*)

Nous considérons le problème de Cauchy pour l'équation de Schrödinger non linéaire sur un domaine du plan avec des conditions aux limites de Dirichlet. Nous prouvons que le problème est bien posé et qu'il existe une solution globale pour une non linéarité polynomiale défocalisante. La preuve repose sur une inégalité de Strichartz généralisée sur des variétés munies d'une métrique de Lipschitz.

1. Introduction

Let Ω be a compact regular domain of $\mathbb{R}^d$, where $d = 2, 3$. The problem we are interested in is the Dirichlet problem for the semilinear Schrödinger equation

Texte reçu le 10 février 2006, révisé le 12 février 2007

Ramona Anton, Université Paris Sud, Mathématiques, Bât 425, 91405 Orsay Cedex (France) • *E-mail :* `Ramona.anton@math.u-psud.fr`

2000 Mathematics Subject Classification. — 35Q55, 35Bxx, 81Q20.

Key words and phrases. — Nonlinear Schrödinger, dispersive equations, Lipschitz metric.

$$
\begin{cases} i\partial_t u + \triangle u = |u|^\beta u, & \text{on } \mathbb{R}\times\Omega, \\ u_{|t=0} = u_0, & \text{on } \Omega, \\ u_{|\mathbb{R}\times\partial\Omega} = 0. \end{cases} \tag{1}
$$

More precisely we are interested in proving global existence results in the energy space $H_0^1(\Omega)$ and this will be done for $d = 2$.

This problem has been extensively study in the case of $\Omega = \mathbb{R}^d$. Note that the sign of the nonlinearity gives an a priori bound of the H^1 norm of the flow and thus allows to prove existence of weak solutions in $C(\mathbb{R}, H_w^1(\mathbb{R}^d))$. The existence of global strong solution is more difficult. One of the main ingredient to address this difficulty is the Strichartz inequality for the linear flow $\mathrm{e}^{it\triangle}$. It can be seen as an improvement of the Sobolev imbedding $H^1 \hookrightarrow L^q$ and the price to pay is an average in time rather than a pointwise information. In $\mathbb{R}^d$, the Strichartz inequality reads as follows: for (p, q) an admissible pair in dimension d and $u_0 \in L^2$

$$
\left\| \mathrm{e}^{it\triangle} u_0 \right\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^d))} \leq c \left\| u_0 \right\|_{L^2}.
$$

Let us recall the definition of an admissible pair.

Definition 1. — A pair (p, q) is called *admissible in dimension d* if

$$
p \geq 2, \quad (p, q, d) \neq (2, \infty, 2) \quad \text{and} \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}\cdot
$$

In 1977 Strichartz [23] proves the particular case $p = q$,

$$
\left\| \mathrm{e}^{i\cdot\triangle} u_0 \right\|_{L^{2+4/d}(\mathbb{R}\times\mathbb{R}^d)} \leq c \left\| u_0 \right\|_{L^2}.
$$

This was generalized by Ginibre and Velo [12], [13] in 1985 for $L_t^p L_x^q$ norm with p and q that satisfy the admissibility condition with $p > 2$ and by Keel and Tao [16] in 1998 for the endpoint $p = 2$ and $q = 2d/(d-2)$. Extension to non homogeneous equation is due to Yajima [28] in 1987 and Cazenave and Weissler [10] in 1988: for (p_1, q_1) and (p_2, q_2) admissible pairs and f in $L^{p_2'}([0,T], L^{q_2'}(\mathbb{R}^d))$ the solution of the non homogeneous equation

$$
i\partial_t u + \triangle u = f, \quad u_{|t=0} = u_0
$$

belongs to $C([0,T], L^2) \cap L^{p_1}([0,T], L^{q_1}(\mathbb{R}^d))$ and satisfies to

$$
\|u\|_{L^{p_1}([0,T], L^{q_1}(\mathbb{R}^d))} \leq c \|f\|_{L^{p_2'}([0,T], L^{q_2'}(\mathbb{R}^d))}.
$$

A contraction mapping argument and those Strichartz inequalities imply the global existence

THEOREM (see [28], [10]). — *For $d \geq 2$ and $1 \leq \beta < 4/(d-2)$ there exists a unique solution*

$$u \in C\big(\mathbb{R}, H^1(\mathbb{R}^d)\big) \cap L^{p_1}_{\mathrm{loc}}\big(\mathbb{R}, W^{1,q_1}(\mathbb{R}^d)\big),$$

for each (p,q) admissible pair in dimension d, of the equation

$$i\partial_t u + \triangle u = |u|^\beta u, \quad u_{|t=0} = u_0.$$

For $\Omega \neq \mathbb{R}^d$ much less is known. In the case of the tori $\mathbb{T}^d$, $d = 2,3$, Bourgain [4] proved global existence result using less stringent dispersive estimates. In the case of a boundaryless compact manifold Burq, Gérard and Tzvetkov [6] proved Strichartz inequalities with loss of derivatives and showed that those losses are in some specific geometries unavoidable.

In the case of domains of $\mathbb{R}^2$ and for cubic equations previous results were proved by Brezis and Gallouet [5] in 1980 and Vladimirov [27] in 1984.

THEOREM (see [5] and [27]). — *For $u_0 \in H^1_0(\Omega)$ there exists a unique solution $u \in C(\mathbb{R}, H^1_0(\Omega))$ of the cubic nonlinear equation*

$$i\partial_t u + \triangle u = |u|^2 u; \ \text{on } \mathbb{R}\times\Omega, \ u_{|t=0} = u_0, \ \text{on } \Omega.$$

Moreover, if $u_0 \in H^1_0(\Omega) \cap H^2(\Omega)$ then

$$u \in C\big(\mathbb{R}, H^1_0(\Omega) \cap H^2(\Omega)\big) \cap C^1\big(\mathbb{R}, L^2(\Omega)\big).$$

The main ingredients of the proof are the logarithmic inequalities

$$\text{(B.-G.)} \qquad \|u\|_{L^\infty} \leq C\,\|u\|_{H^1}\,\big(1 + \log\big(2 + \|u\|_{H^2}/\|u\|_{H^1}\big)\big)^{1/2},$$

$$\text{(V.)} \qquad \forall p < \infty, \quad \|u\|_{L^p} \leq c\sqrt{p}\,\|u\|_{H^1}\,.$$

The methods used in this proof do not give us informations about nonlinearities stronger than cubic. Note that even in this cubic case, the proof did not yield the Lipschitz continuity on the energy space, which is a consequence of Strichartz estimate in the case of $\Omega = \mathbb{R}^d$.

In this article we prove a generalized Strichartz inequality for the Schrödinger flow $\mathrm{e}^{it\triangle}$, where $\triangle$ is the Laplace operator on domains of $\mathbb{R}^d$, $d = 2,3$. Let us introduce the following notation: for every $s \in [0,1]$, we denote by $H^s_D(\Omega)$ the domain of the operator $(-\triangle_D)^{s/2}$ in $L^2(\Omega)$, where $\triangle_D$ is the Dirichlet Laplacian. We refer to Section 3 for more details. We translate the problem on the domain into a problem on a boundaryless Riemannian manifold by doing a mirror reflection of the domain and identifying the points on the boundary. We make also an even reflection of the coefficients of the metric over the boundary in normal coordinates. Thus we obtain a metric with Lipschitz coefficients.

We combine ideas from [2] (see also [25]) on regularizing the metric with a semiclassical analysis of the flow like in [6] and obtain the following Strichartz inequality (with loss of derivatives) in a general context: M is a compact (or flat outside a compact set) Riemannian manifold of dimension $d = 2, 3$, endowed with a Lipschitz metric G.

THEOREM 1.1. — *Let I be a finite time interval, (p, q) an admissible pair in dimension $d = 2, 3$. Let $\epsilon > 0$ be an arbitrarily small constant. Then there exists a constant $c(p, I) > 0$ such that, for all $v_0 \in H^{3/2p+\epsilon}(M)$,*

$$\left\| \mathrm{e}^{it\triangle_G} v_0 \right\|_{L^p(I, L^q(M))} \leq c(p, I) \left\| v_0 \right\|_{H^{3/2p+\epsilon}} . \tag{2}$$

For a compact C^2 perturbation of the Laplacian with nontrapping condition, G. Staffilani and D. Tataru [22] proved Strichartz inequalities without loss of derivatives. In $1D$ with BV metric similar results were obtain by V. Banica [3], D. Salort [19] and N. Burq and F. Planchon [7]. C. Castro and E. Zuazua [8] proved that Strichartz estimates (even with loss of derivatives) fail for metrics only $C^{0,\alpha}$ with $0 \leq \alpha < 1$. Our result shows a Strichartz inequality with loss of derivatives for $C^{0,1}$ metric.

Applying Theorem 1.1 for M the reflection of Ω and for G the reflected metric, we deduce the following theorem.

THEOREM 1.2. — *Let (p, q) be an admissible pair in dimension $d = 2$ or 3 and I a finite time interval. Then for every $\epsilon > 0$, there exists a constant $c(p, I, \epsilon) > 0$ such that for any $u_0 \in H_D^{3/2p+\epsilon}(\Omega)$ and $f \in L^1(I, H_D^{3/2p+\epsilon}(\Omega))$,*

$$\begin{cases} \left\| \mathrm{e}^{it\triangle} u_0 \right\|_{L^p(I, L^q(\Omega))} \leq c(p, I, \epsilon) \left\| u_0 \right\|_{H^{3/2p+\epsilon}(\Omega)} , \\ \left\| \displaystyle\int_0^t \mathrm{e}^{i(t-\tau)\triangle} f(\tau) \mathrm{d}\tau \right\|_{L^p(I, L^q(\Omega))} \leq c(p, I, \epsilon) \left\| f \right\|_{L^1(I, H^{3/2p+\epsilon}(\Omega))} . \end{cases} \tag{3}$$

This inequality gives us a gain of $1/2p - \epsilon$ derivatives with respect to the Sobolev imbedding. Compared with the Strichartz inequality obtained in the case of boundaryless Riemannian compact manifolds in [6] we have a supplementary loss of $1/2p + \epsilon$.

One could ask about the optimality of those estimates. An usual way of checking optimality is testing the estimates for solutions of the Schrödinger flow with initial data eigenfunctions of the Laplacian. This yields some $L^2 \to L^q$ estimates for the eigenfunctions and we look for the optimality of those ones. We refer to some recent work of H. Smith and C. Sogge [20] where they prove $L^2 \to L^q$ estimates for spectral clusters on regular compact domains of $\mathbb{R}^d$, $d \geq 2$. Compared to those estimates, the Strichartz estimate we obtain is not optimal. Nevertheless, it has the advantage of being true for all solutions of the linear Schrödinger equation, not only those with initial data an eigenfunction. And it allows us to prove local and global existence results for the solutions of (1) in dimension $d = 2$.