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ABOUT THE GENERATING FUNCTION OF A LEFT BOUNDED INTEGER-VALUED RANDOM VARIABLE

by Charles Delorme & Jean-Marc Rinkel

Abstract. — We give a relation between the sign of the mean of an integer-valued, left bounded, random variable X and the number of zeros of $1-\Phi(z)$ inside the unit disk, where Φ is the generating function of X, under some mild conditions

Résumé (*Fonction génératrice d'une variable aléatoire à valeurs entières minorées*)
Nous donnons une relation entre le signe de l'espérance d'une variable aléatoire à valeurs entières minorées et le nombre des zéros de $1-\Phi(z)$ dans le disque unité où Φ est la fonction génératrice de X et ce sous des conditions peu exigeantes.

1. Introduction

We consider an integer-valued, left bounded, random variable X. Its generating function Φ is $z \mapsto \sum_{k\in\mathbb{Z}} c_k z^k$, with $c_k = \mathbf{P}(X=k)$. If X is left bounded, with $\inf(X) = -p$, we have $\Phi(z) = \sum_{k\geq -p} c_k z^k$. The object of this paper is the evaluation of the number of zeros inside the unit disc of the function $z \mapsto 1-\Phi(z)$. A motivation for counting these zeros is the exact and asymptotical estimation of the potentials related to random walks on an interval $[0,N]$ defined by $S_n = S_0 + X_1 + \cdots + X_n$ where X_i's are iid with integer left bounded

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distribution and S_0 is an integer valued random variable distributed on the interval $[0, N]$. These potentials are given by the entries of the inverse of Toeplitz matrices (see [3]). But the computation of the inverse uses the factorisation of the function $\theta \mapsto 1 - \Phi(e^{i\theta})$, called the symbol of the Toeplitz operator. One can read [2] for a detailed account. A theorem summarises the results obtained in this paper.

THEOREM 1. — *Let X be a integer valued left bounded random variable, $z \mapsto \Phi(z) = \sum_{k\geq -p}^{\infty} c_k z^k$ its generating function, where $p = -\inf(X) > 0$. We denote by ρ the convergence radius of the series $\sum_{k\geq -p} c_k z^k$, ζ the number of zeros (taking multiplicities into account) inside the unit disc of $z \mapsto 1-\Phi(z)$, d the GCD of the k's such that $c_k > 0$, in other words d is the period of X. Then*

$$\zeta = \begin{cases} -\inf(X) \text{ if } E(X) \in (0, \infty] \\ -d - \inf(X) \text{ if } \begin{cases} E(X) < 0 \text{ or} \\ E(X) = 0 \text{ and } X \text{ has a variance.} \end{cases} \end{cases}$$

If $\rho = 1$ and $E(X) = 0$, then $\zeta \leq -d - \inf(X)$.

If $p \leq 0$, then obviously $1 - \Phi$ has no zero inside the unit disk (except for the very uninteresting case where $c_0 = 1$), because the absolute value of a sum is at most the sum of the absolute value of its terms.

The random variable X is aperiodic precisely when this $d = 1$.

CONJECTURE 1. — *With the hypotheses and notations of Theorem 1, we have $\zeta = -d - \inf(X)$ even if $E(X) = 0$.*

In order to support the conjecture let us notice that for $E(X) = 0$, some other classes of functions give again $p - d$ as the number of zeros. For example, if $p = 1$ and $d = 1$ (obviously) or $p = 2$: indeed there is at least one zero on the interval $(-1, 0)$ when p is even since $\Phi(-1) < 1$ (see lemma 1) and $\lim_{x\to 0^-} \Phi(x) = +\infty$).

REMARK 1. — *It suffices to prove the theorem (or the conjecture) in the case $d = 1$ to obtain the general one. In fact, this is obtained by noticing that $\Phi(z) = \phi(z^d)$ with ϕ the generating function of the integer-valued and aperiodic random variable X/d. Denoting by ζ_1 the number of zeros (with multiplicities) inside the unit disc of $1 - \phi$, we have then $\zeta = d\zeta_1$. We deduce that $\zeta = p = -\inf(X)$ when $\zeta_1 = p/d = -\inf(X/d)$ and $\zeta = -d + p = -d - \inf(X)$ when $\zeta_1 = -1 + p/d = -1 - \inf(X/d)$.*

From now on, we will consider only the case $d = 1$.

2. Some lemmas

We will use two well-known propositions

LEMMA 1 (Generalized triangle inequality). — *Let a_i, $0 \le i$ be an absolutely convergent series of complex numbers. Then $|\sum_{i=0}^{\infty} a_i| \le \sum_{i=0}^{\infty} |a_i|$. Moreover, the equality occurs only if the non-null terms have the same argument. Of course, this common argument is also the argument of the sum itself.*

LEMMA 2 (Rouché's theorem [1, p. 13]). — *Let D a domain in the complex plane, let C be a simple closed curve and let f, g be two functions, meromorphic on D, having no pole on C, such that $|g(z)| < |f(z)|$ for all z of C. Then $f + g$ and f have the same total number of zeros and poles, counted with multiplicities, inside C.*

LEMMA 3. — *Let p be a positive integer and c_k, $k \ge p$ be a sequence of non-negative reals, such that*

- $c_{-p} > 0$,
- $\sum_{k \ge -p} c_k = 1$,
- *the GCD of the k's such that $c_k > 0$ is 1.*

Then the series $\Phi(z) = \sum_{k \ge -p} c_k z^k$ converges on the punctured disk $0 < |z| \le 1$, its sum is continuous on that disk, and meromorphic on the open disk $|z| < 1$. The convergence of the series to its sum is uniform. The only point on the circle $|z| = 1$ where $\Phi(z) = 1$ is 1.

Proof. — The remainder of the series is $\sum_{k \ge N} c_k z^k$ for $N \ge 0$ is at most $\sum_{k > N} c_k$ since $|z^k| \le 1$ and $c_k \ge 0$. Since the series converges at the point $z = 1$, we have proven the uniform convergence. The continuity of Φ and its being meromorphic are classical consequences of this uniform convergence.

Let us consider the set S of integers such that $c_k > 0$. The equality $\sum_S c_k z^k = 1 = \sum_S c_k$ with $|z| = 1$ and $c_k \ge 0$ imposes that $z^k = 1$ for all k's in S, due to the generalized triangle inequality. Since the GCD of these k's is 1, there is a finite part T of S and integers $\beta_k, k \in T$ such that $\sum_{k \in T} \beta_k k = 1$ Then

$$z = z^{\Sigma_{k \in T} \beta_k k} = \prod_{k \in T} (z^k)^{\beta_k} = 1$$

and 1 is the only point on the unit circle where $\Phi = 1$. $\square$

LEMMA 4. — *The restriction of $\Phi(z) = \sum_{k \ge -p} c_k z^k$ to the real interval $(0, 1]$ either takes value 1 at point 1 only or takes value 1 at an other point r inside the interval $(0, 1)$. In the first case, X has a mean $E(X) = \sum_{k \ge -p} k c_k$ and $E(X) \le 0$. In the second case it has a positive mean or the series $\sum_{k \ge -p} k c_k$ diverges to $+\infty$, and $\Phi(x) < 1$ on the interval $(r, 1)$*

Proof. — This results from the observation that the restriction of Φ is continuous, strictly convex, and $\Phi(x) \to \infty$ when $x \to 0^+$. $\square$

3. The case with no mean value or positive mean value

In this case, let r be the point in $(0,1)$ where $\Phi(r) = 1$. For each radius s such that $r < s < 1$, we see that on the circle centered at 0 with radius s we have $|\Phi(z)| \leq \Phi(s) < 1$. This circle is inside the domain where Φ is meromorphic. By Rouché's theorem, we can assert that the number of zeros and poles counted with their order inside the circle of radius s is the same for the function 1 and the function $1 - \Phi$. This common number is of course 0, since the function 1 has neither zeros nor poles! Since $1 - \Phi$ has just a pole of order p, it has zeros whose orders sum up to p inside the disk of radius s. Since the open unit disk is the union of the open disks of radiuses $s < 1$, it contains p zeros counted with their order.

We moreover see that r is the zero of $1-\Phi$ with largest absolute value inside the unit disk, and it is the only zero of absolute value r, again an application of the generalized triangle inequality. It is simple, since $\Phi'(r) < 0$.

4. The case of negative mean value and convergence radius > 1

A proof in the same vein allows also to estimate the number of zeros when the convergence radius is > 1 and the mean value < 0. Since the derivative of Φ at 1 is < 0, there is a real r, smaller than the convergence radius, with $1 < r$ such that $\Phi(x) < 1$ on the interval $(1, r]$. With the same trick as in the former section, the circles of radius x with $1 < x < r$ do not contain zeros of $1-\Phi$, because $|\Phi(z)| \leq \Phi(x) < 1$ for $|z| = x$. Thus the disk of radius r contains only zeros of $1 - \Phi$ of absolute value ≤ 1. According to Rouché's theorem, the number of zeros and poles (counted with their order) of 1 and $1 - \Phi$ inside the disk of radius x coincide. This number is 0. Because of the pole of order p at 0, there are p zeros (counted with their order) inside the disk of radius r, one of them is the simple zero at 1, and it is the only one with absolute value 1.

Thus the open disk of radius 1 contains $p - 1$ zeros of $1 - \Phi$ (counted with their order).

5. The case of mean value ≤ 0 and convergence radius 1: an inequality

Since the function $1 - \Phi$ is meromorphic on the open unit disk, its zeros inside the open disk are isolated. However they may accumulate towards the unit circle. Since the function is continuous on the punctured disk, the only possible limit of a sequence of zeros of $1 - \Phi$ is the point 1.