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ON THE IWAHORI WEYL GROUP

by Timo Richarz

Abstract. — Let F be a discretely valued complete field with valuation ring $\mathcal{O}_F$ and perfect residue field k of cohomological dimension ≤ 1. In this paper, we generalize the Bruhat decomposition in Bruhat and Tits [3] from the case of simply connected F-groups to the case of arbitrary connected reductive F-groups. If k is algebraically closed, Haines and Rapoport [4] define the Iwahori-Weyl group, and use it to solve this problem. Here we define the Iwahori-Weyl group in general, and relate our definition of the Iwahori-Weyl group to that of [4].

Let F be a discretely valued complete field with valuation ring $\mathcal{O}_F$ and perfect residue field k of cohomological dimension ≤ 1. In this paper, we generalize the Bruhat decomposition in Bruhat and Tits [3] from the case of simply connected F-groups to the case of arbitrary connected reductive F-groups. If k is algebraically closed, Haines and Rapoport [4] define the Iwahori-Weyl group, and use it to solve this problem. Here we define the Iwahori-Weyl group in general, and relate our definition of the Iwahori-Weyl group to that of [4]. Furthermore, we study the length function on the Iwahori-Weyl group, and use it to determine the number of points in a Bruhat cell, when k is a finite field. Except for Lemma 1.3 below, the results are independent of [4], and are directly based on the work of Bruhat and Tits [2], [3].

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Let $\bar{F}$ be the completion of a separable closure of F. Let $\breve{F}$ be the completion of the maximal unramified subextension with valuation ring $\mathcal{O}_{\breve{F}}$ and residue field $\bar{k}$. Let $I = \mathrm{Gal}(\bar{F}/\breve{F})$ be the inertia group of $\breve{F}$, and let $\Sigma = \mathrm{Gal}(\breve{F}/F)$.

Let G be a connected reductive group over F, and denote by $\mathscr{B} = \mathscr{B}(G, F)$ the enlarged Bruhat-Tits building. Fix a maximal F-split torus A. Let $\mathscr{A} = \mathscr{A}(G, A, F)$ be the apartment of $\mathscr{B}$ corresponding to A.

1.1. Definition of the Iwahori-Weyl group. — Let $M = Z_G(A)$ be the centralizer of A, an anisotropic group, and let $N = N_G(A)$ be the normalizer of A. Denote by $W_0 = N(F)/M(F)$ the relative Weyl group.

DEFINITION 1.1. — i) The *Iwahori-Weyl group* $W = W(G, A, F)$ is the group

$$W \overset{\mathrm{def}}{=} N(F)/M_1,$$

where M_1 is the unique parahoric subgroup of $M(F)$.

ii) Let $\mathfrak{a} \subset \mathscr{A}$ be a facet and $P_{\mathfrak{a}}$ the associated parahoric subgroup. The subgroup $W_{\mathfrak{a}}$ of the Iwahori-Weyl group corresponding to $\mathfrak{a}$ is the group

$$W_{\mathfrak{a}} \overset{\mathrm{def}}{=} P_{\mathfrak{a}} \cap N(F)/M_1.$$

The group $N(F)$ operates on $\mathscr{A}$ by affine transformations

$$\nu : N(F) \longrightarrow \mathrm{Aff}(\mathscr{A}). \tag{1.1}$$

The kernel $\ker(\nu)$ is the unique maximal compact subgroup of $M(F)$ and contains the compact group M_1. Hence, the morphism (1.1) induces an action of W on $\mathscr{A}$.

Let G_1 be the subgroup of $G(F)$ generated by all parahoric subgroups, and define $N_1 = G_1 \cap N(F)$. Fix an alcove $\mathfrak{a}_C \subset \mathscr{A}$, and denote by B the associated Iwahori subgroup. Let $\mathbb{S}$ be the set of simple reflections at the walls of $\mathfrak{a}_C$. By Bruhat and Tits [3, Prop. 5.2.12], the quadruple

$$(G_1, B, N_1, \mathbb{S}) \tag{1.2}$$

is a (double) Tits system with affine Weyl group $W_{\mathrm{af}} = N_1/N_1 \cap B$, and the inclusion $G_1 \subset G(K)$ is B-N-adapted of connected type.

LEMMA 1.2. — *i) There is an equality* $N_1 \cap B = M_1$.

ii) The inclusion $N(F) \subset G(F)$ *induces a group isomorphism* $N(F)/N_1 \xrightarrow{\cong} G(F)/G_1$.

Proof. — The group $N_1 \cap B$ operates trivially on $\mathscr{A}$ and so is contained in $\ker(\nu) \subset M(F)$. In particular, $N_1 \cap B = M(F) \cap B$. But $M(F) \cap B$ is a parahoric subgroup of $M(F)$ and therefore equal to M_1.
The group morphism $N(F)/N_1 \to G(F)/G_1$ is injective by definition. We have to show that $G(F) = N(F) \cdot G_1$. This follows from the fact that the inclusion $G_1 \subset G(F)$ is B-N-adapted, cf. [2, 4.1.2]. □

Kottwitz defines in [5, §7] a surjective group morphism

$$\kappa_G : G(F) \longrightarrow X^*(Z(\hat{G})^I)^\Sigma. \tag{1.3}$$

Note that in [*loc. cit.*] the residue field k is assumed to be finite, but the arguments extend to the general case.

LEMMA 1.3. — *There is an equality $G_1 = \ker(\kappa_G)$ as subgroups of $G(F)$.*

Proof. — For any facet $\mathfrak{a}$, let $\mathrm{Fix}(\mathfrak{a})$ be the subgroup of $G(F)$ which fixes $\mathfrak{a}$ pointwise. The intersection $\mathrm{Fix}(\mathfrak{a}) \cap \ker(\kappa_G)$ is the parahoric subgroup associated to $\mathfrak{a}$, cf. [4, Proposition 3]. This implies $G_1 \subset \ker(\kappa_G)$. For any facet $\mathfrak{a}$, let $\mathrm{Stab}(\mathfrak{a})$ be the subgroup of $G(F)$ which stabilizes $\mathfrak{a}$. Fix an alcove $\mathfrak{a}_C$. There is an equality

$$\mathrm{Fix}(\mathfrak{a}_C) \cap G_1 = \mathrm{Stab}(\mathfrak{a}_C) \cap G_1, \tag{1.4}$$

and (1.4) holds with G_1 replaced by $\ker(\kappa_G)$, cf. [4, Lemma 17]. Assume that the inclusion $G_1 \subset \ker(\kappa_G)$ is strict, and let $\tau \in \ker(\kappa_G)\backslash G_1$. By Lemma 1.2 ii), there exists $g \in G_1$ such that $\tau' = \tau \cdot g$ stabilizes $\mathfrak{a}_C$, and hence τ' is an element of the Iwahori subgroup $\mathrm{Stab}(\mathfrak{a}_C) \cap \ker(\kappa_G)$. This is a contradiction, and proves the lemma. □

By Lemma 1.2, there is an exact sequence

$$1 \longrightarrow W_{\mathrm{af}} \longrightarrow W \xrightarrow{\kappa_G} X^*(Z(\hat{G})^I)^\Sigma \longrightarrow 1. \tag{1.5}$$

The stabilizer of the alcove $\mathfrak{a}_C$ in W maps isomorphically onto $X^*(Z(\hat{G})^I)^\Sigma$ and presents W as a semidirect product

$$W = X^*(Z(\hat{G})^I)^\Sigma \ltimes W_{\mathrm{af}}. \tag{1.6}$$

For a facet $\mathfrak{a}$ contained in the closure of $\mathfrak{a}_C$, the group $W_\mathfrak{a}$ is the parabolic subgroup of W_{af} generated by the reflections at the walls of $\mathfrak{a}_C$ which contain $\mathfrak{a}$.

THEOREM 1.4. — *Let $\mathfrak{a}$ (resp. $\mathfrak{a}'$) be a facet contained in the closure of $\mathfrak{a}_C$, and let $P_\mathfrak{a}$ (resp. $P_{\mathfrak{a}'}$) be the associated parahoric subgroup. There is a bijection*

$$\begin{aligned} W_\mathfrak{a}\backslash W/W_{\mathfrak{a}'} &\xrightarrow{\cong} P_\mathfrak{a}\backslash G(F)/P_{\mathfrak{a}'} \\ W_\mathfrak{a}\, w\, W_{\mathfrak{a}'} &\longmapsto P_\mathfrak{a}\, n_w\, P_{\mathfrak{a}'}, \end{aligned}$$

where n_w denotes a representative of w in $N(F)$.