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GRADINGS ON LIE ALGEBRAS, SYSTOLIC GROWTH, AND COHOPFIAN PROPERTIES OF NILPOTENT GROUPS

BY YVES CORNULIER

ABSTRACT. — We study the existence of various types of gradings on Lie algebras, such as Carnot gradings or gradings in positive integers, and prove that the existence of such gradings is invariant under extensions of scalars.

As an application, we prove that if Γ is a finitely generated nilpotent group, its systolic growth is asymptotically equivalent to its word growth if and only if the Malcev completion of Γ is Carnot.

We also characterize when Γ is non-cohopfian, in terms of the existence of a non-trivial grading in non-negative integers, and deduce that this property only depends on its real (or even complex) Malcev completion.

RÉSUMÉ (*Graduations sur les algèbres de Lie, croissance systolique, et propriétés cohopfiennes des groupes nilpotents*)

On s'intéresse à l'existence de différentes sortes de graduations sur les algèbres de Lie, notamment les graduations Carnot ou en entiers positifs. On démontre notamment que l'existence de telles graduations est invariant par extension des scalaires.

On utilise ce résultat pour établir que si Γ est un groupe nilpotent de type fini, sa croissance systolique est asymptotiquement équivalente à sa croissance des mots si et seulement si la complétion de Malcev de Γ est Carnot.

On caractérise également quand Γ est non-cohopfien, en termes de l'existence d'une graduation non triviale en entiers positifs, et on en déduit que cette propriété ne dépend que de sa complétion de Malcev réelle.

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1. Introduction

The purpose of this paper is twofold: to discuss the existence of certain kinds of gradings on Lie algebras (and some more general algebras), and to apply the results to the study of several aspects of finitely generated nilpotent groups.

1.1. Cohopfian properties and systolic growth.

— This paper will study some properties in the case of finitely generated nilpotent groups. In this subsection, we introduce these properties in general.

1.1.1. *Systolic growth.* — Let Γ be a finitely generated group, and endow it with the word metric with respect to some finite generating subset S. If $\Lambda \subset \Gamma$, define its systole $\mathrm{sys}_S(\Lambda)$ to be $\inf\{|g|_S : g \in \Lambda \smallsetminus \{1\}\}$ (which is $+\infty$ in case $\Lambda = \{1\}$). Define, following [20] its *systolic growth* as the function $\sigma_{\Gamma,S}$ mapping n to the smallest index of a subgroup of systole $\geq n$ (hence $+\infty$ if there is no such subgroup). Note that Γ is residually finite if and only if $\sigma_{\Gamma,S}(n) < \infty$ for all n, and a standard argument shows that the asymptotic behavior (in the usual sense of growth of groups, see §6) of $\sigma_{\Gamma,S}$ does not depend on the choice of S; hence we call it the systolic growth of Γ. It is obviously asymptotically bounded below by the growth (precisely, $\sigma_{\Gamma,S}(2n+1) \geq b_{\Gamma,S}(n)$, where $b_{\Gamma,S}(n)$ is the cardinal of the n-ball). It is easy to see that Γ and its finite index subgroups have asymptotically equivalent systolic growth.

It is natural to wonder when the growth and systolic growth are equivalent. Gromov [20, p.334] provides a simple argument, based on congruence subgroups, showing that finitely generated subgroups of $\mathrm{GL}_d(\mathbf{Q})$ have at most exponential systolic growth (although he states a less general fact). Bou-Rabee and the author [6] actually prove that all finitely generated linear groups (i.e., with a faithful finite-dimensional linear representation over some field) have at most exponential systolic growth, and hence exactly exponential systolic growth when the growth is exponential. For finitely generated linear groups, this thus reduces the question of equivalence of growth and systolic growth to virtually nilpotent groups.

Remark 1.1. — A notion closely related to systolic growth was introduced by Bou-Rabee and McReynolds [7] (apparently independently of [20]), defining the *residual girth* of a group in the same way as the systolic growth above, but restricting to *normal* finite index subgroups. If we denote by $\sigma^{\lhd}_{\Gamma,S}(n)$ the resulting function, we obviously have

$$\sigma_{\Gamma,S} \leq \sigma^{\lhd}_{\Gamma,S} \leq \sigma_{\Gamma,S}!.$$

The examples in [8] show that $\sigma_{\Gamma,S}$, for finitely generated residually finite groups, can be arbitrary large.

On the other hand, there is an exponential upper bound for the residual girth of finitely generated linear groups [6].

Besides, we can define one more notion: if $\Lambda \subset \Gamma$, define its *normal systole* $\mathrm{sys}_S^{\triangleleft}(\Lambda)$ as the infimum of $|g|_S$, when g ranges over Γ-*conjugates* of elements in $\Lambda \smallsetminus \{1\}$. Note that it has a geometric interpretation: let $\mathscr{G}(\Gamma, S)$ be the Cayley graph of Γ with respect to S. While $\mathrm{sys}_S(\Lambda)$ is the length of the smallest non-trivial based combinatorial loop in the quotient $\Lambda \backslash \mathscr{G}(\Gamma, S)$ (where non-trivial means it does not lift to a loop in $\mathscr{G}(\Gamma, S)$), the normal systole $\mathrm{sys}_S^{\triangleleft}(\Lambda)$ is the length of the smallest non-trivial combinatorial loop (not necessarily based); of course when Λ is normal, its normal systole equals its systole. We can then define the *uniform systolic growth* of Γ as the function $\sigma^u_{\Gamma,S}$ mapping n to the smallest index of a subgroup of normal systole $\geq n$. Thus clearly we have

$$\sigma_{\Gamma,S} \leq \sigma^u_{\Gamma,S} \leq \sigma^{\triangleleft}_{\Gamma,S}.$$

I do not know examples for which the uniform systolic growth is not equivalent to the systolic growth; on the other hand simple examples show that it can fail to be equivalent to the residual girth, see Remark 1.9.

1.1.2. *Cohopfian properties.* — Recall that a group is *non-cohopfian* if it admits a non-surjective injective endomorphism, and *cohopfian* otherwise. Let us say that a group Γ is *dis-cohopfian* if it admits an injective endomorphism ϕ such that $\bigcap_{n \geq 0} \phi^n(\Gamma) = \{1\}$; such ϕ is called a dis-cohopf endomorphism. It appears, for a nontrivial group, as a strong negation of being cohopfian.

Let us consider an intermediate notion: we say that a group Γ is *weakly dis-cohopfian* if it admits a sequence of subgroups (Γ_n), all isomorphic to Γ, with Γ_{n+1} contained in Γ_n for all n, and $\bigcap_n \Gamma_n = \{1\}$. This is implied by dis-cohopfian, and implies, for a nontrivial group, non-cohopfian.

Example 1.2. — The group $\mathbf{Z}$ is dis-cohopfian. Slightly less trivially, the infinite dihedral group is dis-cohopfian. Actually, every nontrivial free product $A*B$ is dis-cohopfian (that non-trivial free products are non-cohopfian is well-known; whether they are always dis-cohopfian was asked to me by K. Bou-Rabee): indeed, fix nontrivial elements $a_0 \in A$, $b_0 \in B$ and consider the endomorphism ϕ defined by $a \mapsto a_0 b_0 a b_0^{-1} a_0^{-1}$ for $a \in A$ and $b \mapsto b_0 a_0 b a_0^{-1} b_0^{-1}$. Let S be the generating set $A \cup B$ for $A * B$, and $|\cdot|$ the corresponding word length. Then this endomorphism formally maps reduced words to reduced words, and hence $|\phi(x)| = 5|x|$ for all x; thus $|\phi^n(x)| = 5^n|x|$ for all x and it follows that $\bigcap_{n \geq 0} \mathrm{Im}(\phi^n) = \{1\}$.

The group $\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ is non-cohopfian but not weakly dis-cohopfian. Examples of groups that are weakly dis-cohopfian but not dis-cohopfian will be provided in Example 5.16. These are the first such examples among finitely