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An introduction to expander graphs

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26

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… à l'expansion de mon cœur refoulé s'ouvrirent aussitôt des espaces infinis.

M. Proust, À l'ombre des jeunes filles en fleurs
(deuxième partie, "Noms de Pays : le Pays")

He walked as if on air, and the whole soul had obviously expanded,
like a bath sponge placed in water.

P.G. Wodehouse, Joy in the Morning
(Chapter 16)

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PREFACE

The goal of this book is to give an introduction to *expander graphs* and their applications. It is therefore related to the books of Lubotzky [**78**] (and his Colloquium Lectures [**79**]), of Sarnak [**100**], and of Tao [**109**], and to the detailed survey of Hoory, Linial and Wigderson [**54**]. Each of these is a wonderful source of information, but we hope that some readers will also find interest in special features of this new text. I hope in particular that the discussion of the basic formalism of graphs and of expansion in graphs, which is more detailed than is usual, will be helpful for many mathematicians. Of course, others might find it just pedantic; for these readers, maybe the variety of constructions of expander graphs, and of applications (some of which have not been discussed in previous books) will be a redeeming feature.

The first version of these notes were prepared in parallel with a course that I taught at ETH Zürich during the Fall Semester 2011. I thank all the people who attended the course for their remarks and interest, and corrections, in particular E. Baur, P.-O. Dehaye, O. Dinai, T. Holenstein, B. Löffel, L. Soldo and P. Ziegler. Also, many thanks to R. Pink for discussing various points concerning non-concentration inequalities and especially for helping in the proof of the special case needed to finish the proof of Helfgott's theorem in Chapter 6.

The text was continued (including both corrections and changes and the addition of some material) for a short course at TU Graz during the Spring Semester 2013. Many thanks to R. Tichy and C. Elsholtz for the invitation to give this course.

The final version arose also from teaching various minicourses in Neuchâtel ("Expanders everywhere!"), Lyon ("Colloque Jeunes Chercheurs en Théorie des Nombres"), and during the Ventotene 2015 conference "Manifolds and groups". I thank the respective organizers (A. Valette and A. Khukro for the first; L. Berger, M. Carrizosa, W. Nizioł, E. Royer and S. Rozensztajn for the second; S. Francaviglia, R. Frigerio, A. Iozzi, K. Juschenko, G. Mondello and M. Sageev for the last) for inviting me, and especially A. Iozzi for the last one, which was especially enjoyable in view of its setting. The last step was teaching a course again in the Spring 2016 at ETH Zürich; thanks to B. Löffel and to J. Volec for their help and corrections at that time.

I also thank M. Burger, J. Ellenberg, C. Hall and A. Valette for many discussions related to expanders (and their applications) over the years, and finally I thank N. Bourbaki for his kind invitation to talk in his seminar about expanders and sieve.

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Zürich, October 2017.

CHAPTER 1

INTRODUCTION AND MOTIVATION

This short chapter is highly informal, and the reader should not worry if parts are not immediately understood, or are somewhat ambiguous: we will come back with fully rigorous definitions of all terms later.

Our goal is to introduce *expander graphs*. The outline is roughly the following: (1) we will explain the definition, or precisely give three definitions and show that they are equivalent; (2) we will then give different proofs of the existence of expanders (which is by no means obvious from the definition!), first the original one (based on probabilistic methods) and then three others (with more or less details); (3) we will then present some of the remarkably varied and surprising applications of expanders, with a focus in "pure" mathematics (this part is to some extent a survey, since explaining from scratch the context in all cases would require too much space).

We begin with a brief informal outline of the definition of expanders, and some of their applications. Hopefully, the reader will be convinced that it is a story worth knowing something about, and turn to the rest of the book...

To start with, graphs seem very intuitive mathematical objects. For the moment, we consider them in this manner, while in the next chapter we will give a formal definition. So we view a graph as a set V of vertices, and a set E of edges joining certain pairs (x, y) of vertices, and we allow the possibility of having multiple edges between x and y, as well as loops joining a vertex x to itself. We visualize graphs geometrically, and think of the edges as ways to go from one vertex to another. For our purpose, these edges are considered to be unoriented. One can then speak of "which vertices can be linked to a given vertex x", or of the distance between two vertices x and y as the length of the shortest sequence of edges starting from x and ending at y.

Graphs enter naturally in many concrete problems as models for real-life objects, possibly using different conventions (e.g., oriented edges). Here are a few examples:

– *[Transport network]* In a given geographical area (a town, a country, or even the earth) one often visualizes the transport possibilities within this area (possibly restricted to certain means of transportation, such as trains, tramways, subways, planes, roads) as a graph. For instance, Figure 1.1 represents the well-known tramway network of Zürich in 2012. This graph has no loop but it has many

FIGURE 1.1. Zürich tramway network (Wikipedia, Author: mateusch, license Creative Commons Attribution-Share Alike 3.0 Unported.)

multiple edges since a number of lines travel in parallel in certain areas.

– *[The brain]* Viewing neurons as vertices and synapses as edges, the brain of an animal is—in a rather rough first approximation—also a graph. To the author's knowledge (gathered from the internet...), the only species for which this graph has been determined in its entirety is the *nematode Caenorhabditis Elegans* (a worm of size approximately 1 millimeter, see [**120**]); this contains 302 neurons and about 8000 synapses. Determining this graph was done by White, Southgate, Thomson, Brenner [**118**] in 1986 (and corrected in 2011 by Varshney, Chen, Paniagua, Hall, Chklovskii [**117**], from which paper the figure is taken).

– *[Relationship graphs]* Given a set of individuals and a relation between them (such as "X is a relative of Y," or "X knows Y," or "X has written a joint paper with Y"), one can draw the corresponding graph. Its *connectedness* properties are often of interest: this leads, for instance, to the well-known Erdős number of a mathematician, which is the distance to the vertex "Paul Erdős" on the collaboration graph joining mathematicians that have a joint paper. Genealogical trees form another example of this type, although the relation "X is a child of Y" is most naturally considered as an oriented edge.

Expander graphs, the subject of these notes, are certain families of graphs, becoming larger and larger, which have the following two competing properties: (1) they are fairly sparse (in terms of number of edges, relative to the number of vertices); (2) yet they are highly connected, and in fact highly "robust," in some sense.

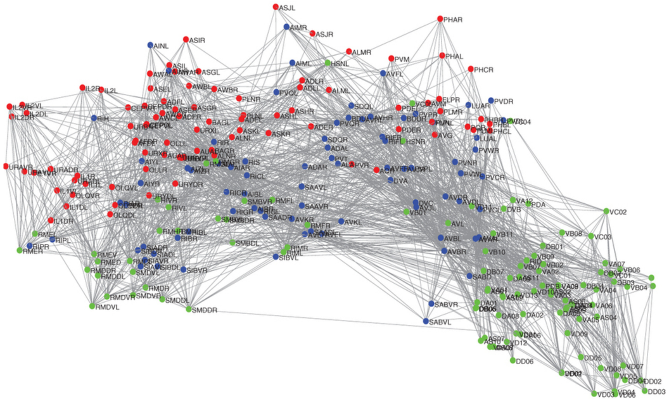

FIGURE 1.2. The nervous system of *Caenorhabditis Elegans*

There are different ways of formalizing these ideas. We assume given a family of finite graphs Γ_n with vertex sets V_n such that the size $|V_n|$ goes to infinity, and we first formalize the condition of sparsity by asking that the *degree* of Γ_n be bounded by some constant $v \geqslant 1$ for all n, i.e., for any n, any vertex $x \in V_n$ has at most v distinct neighbors in V_n. If we think of graphs as objects that might be realized physically (as a communication network with physical links between vertices), with a certain cost associated with each physical edge, this assumption means that increasing the number of vertices (by taking a larger graph Γ_n from our family) will increase the cost linearly with respect to the increase in the number of vertices, since the number of edges of Γ_n is at most $v|V_n|$. Clearly, this sparsity is important if one constructs a tramway network…

The second condition satisfied by expander graphs generalizes the property of *connectedness*, which would simply mean that one can go from any vertex to any other in the graph, by at least some path. One natural strengthening is to ask that such a path is always rather short, which means that the maximal distance in the graph between two points is much smaller than the number of vertices. However, this is not sufficient to define an expander, because a small diameter does not prevent the existence of a "bottleneck" in a graph: even though the graph is connected, there might well exist a rather small subset B of edges such that the graph obtained by removing B (and all edges with at least one extremity in B) is disconnected. To avoid this, one wishes that any subset $V \subset V_n$ of vertices should have many connections with its complement $W = V_n - V$, i.e., there should be many edges linking vertices v and w with $v \in V$ and $w \in W$. Even more precisely, *expanders* are determined by the condition that, for some constant $c > 0$, independent of n, the number of such edges should be at least $c \min(|V|, |W|)$ for all (non-empty) subsets $V \subset V_n$, and for all n.

This definition of sparse, highly connected, robust, families of graphs is obviously quite strong. It is by no means obvious that they exist at all! In fact, as we will see, most