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TOPOLOGICAL PROPERTIES OF RAUZY FRACTALS

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TOPOLOGICAL PROPERTIES OF RAUZY FRACTALS

Anne Siegel, Jörg M. Thuswaldner

Abstract. – Substitutions are combinatorial objects (one replaces a letter by a word) which produce sequences by iteration. They occur in many mathematical fields, roughly as soon as a repetitive process appears. In the present monograph we deal with topological and geometric properties of substitutions, in particular, we study properties of the *Rauzy fractals* associated to substitutions.

To be more precise, let σ be a substitution over the finite alphabet $\mathcal{A}$. We assume that the incidence matrix of σ is primitive and that its dominant eigenvalue is a unit Pisot number (*i.e.*, an algebraic integer greater than one whose norm is equal to one and all of whose Galois conjugates are of modulus strictly smaller than one). It is well-known that one can attach to σ a set $\mathcal{T}$ which is called *central tile* or *Rauzy fractal* of σ. Such a central tile is a compact set that is the closure of its interior and decomposes in a natural way in $n = |\mathcal{A}|$ subtiles $\mathcal{T}(1), \ldots, \mathcal{T}(n)$. The central tile as well as its subtiles are graph directed self-affine sets that often have fractal boundary.

Pisot substitutions and central tiles are of high relevance in several branches of mathematics like tiling theory, spectral theory, Diophantine approximation, the construction of discrete planes and quasicrystals as well as in connection with numeration like generalized continued fractions and radix representations. The questions coming up in all these domains can often be reformulated in terms of questions related to the topology and the geometry of the underlying central tile.

After a thorough survey of important properties of unit Pisot substitutions and their associated Rauzy fractals the present monograph is devoted to the investigation of a variety of topological properties of $\mathcal{T}$ and its subtiles. Our approach is an algorithmic one. In particular, we dwell upon the question whether $\mathcal{T}$ and its subtiles induce a tiling, calculate the Hausdorff dimension of their boundary, give criteria for their connectivity and homeomorphy to a closed disk and derive properties of their fundamental group.

The basic tools for our criteria are several classes of graphs built from the description of the tiles $\mathcal{T}(i)$ $(1 \leq i \leq n)$ as the solution of a graph directed iterated function system and from the structure of the tilings induced by these tiles. These graphs are of interest in their own right. For instance, they can be used to construct the

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boundaries $\partial\mathcal{T}$ as well as $\partial\mathcal{T}(i)$ $(1 \leq i \leq n)$ and all points where two, three or four different tiles of the induced tilings meet.

When working with central tiles in one of the above mentioned contexts it is often useful to know such intersection properties of tiles. In this sense the present monograph also aims at providing tools for "everyday's life" when dealing with topological and geometric properties of substitutions.

Many examples are given throughout the text in order to illustrate our results. Moreover, we give perspectives for further directions of research related to the topics discussed in this monograph.

Résumé **(Propriétés topologiques des fractals de Rauzy).** – Les fractals de Rauzy apparaissent dans diverses branches des mathématiques telles que la théorie des nombres, les systèmes dynamiques, la combinatoire et la théorie des quasi-cristaux. De nombreuses questions font alors intervenir la structure topologique des fractals. Cette monographie propose une étude systématique des propriétés topologiques des fractals de Rauzy. Les premiers chapitres de ce document rappellent les enjeux mathématiques relatifs aux fractals de Rauzy ainsi que les principaux résultats connus à leur sujet. Sont ensuite discutés des propriétés de pavages, de connexité, d'homéomorphisme à un disque, ainsi que le groupe fondamental de ces ensembles. Les méthodes s'appuient sur des résultats en topologie du plan et sur la construction de graphes pour décrire la structure des pavages associés aux fractals. De nombreux exemples caractéristiques sont présentés. Un chapitre final discute des principales perspectives de recherches liées à cette thématique.

CONTENTS

CHAPTER 1

INTRODUCTION

The present monograph deals with topological and geometric properties of substitutions. In this introduction we first emphasize on the great importance of substitutions in many fields of mathematics, theoretical physics and computer science. Already in this first part it becomes evident on several places that geometrical objects like *Rauzy fractals* are intimately related to substitutions and that their topological as well as geometric properties deserve to be studied in order to get information about the underlying substitution. After this general part we give an introductory overview of Rauzy fractals with special emphasis on their topology. We discuss their history and give some details on different ways of their construction. The introduction closes with an outline of the content of this monograph.

1.1. The role of substitutions in several branches of mathematics

A *substitution* (sometimes also called *iterated morphism*) is a combinatorial object which produces sequences by iteration. It is given by a replacement rule of the letters of a finite alphabet by nonempty, finite words over the same alphabet. Thus substitutions define an iteration process on a finite set in a natural way. Therefore, they can be recovered in many fields of mathematics, theoretical physics and computer science whenever repetitive processes or replacement rules occur.

1.1.1. Combinatorics. – In combinatorics on words, since the beginnings of this domain, substitutions have been used in order to exhibit examples of finite words or infinite sequences with very specific or unusual combinatorial properties. The most famous example is the Thue-Morse sequence defined over the two letter alphabet $\{1,2\}$ as follows. Let the *Thue-Morse substitution* $\sigma(1) = 12$, $\sigma(2) = 21$ be given. This substitution admits two infinite fixed points: the first one begins with all iterations $\sigma^m(1)$ $(m \geq 1)$, the second one begins with the words $\sigma^m(2)$ $(m \geq 1)$. The first of these fixed points is called *Thue-Morse sequence*[1]. Thue [**164, 165**] and Morse (see [**92**]) proved for instance that this infinite sequence is overlap-free, meaning that

[1] Note that the second fixed point emerges from the first one just by exchanging the letters.

it contains no subword of the shape *auaua*, where u is a finite word, possibly empty, and a is a single letter (see for instance [**21, 42, 86, 118**] where many other properties of this famous sequence are discussed).

Certain classes of *sturmian sequences* can be defined in terms of substitutions. Sturmian sequences were introduced in the 1940s as sequences having the smallest complexity among all nonperiodic infinite sequences over a two letter alphabet. In particular, the number of their factors of size k is equal to $k+1$ for each $k \in \mathbb{N}$. A famous characterization relates these sequences to geometry. Indeed, sturmian sequences are exactly the cutting sequences of lines in $\mathbb{R}^2$. In particular, first draw grid lines, which are the horizontal and vertical lines through the lattice $\mathbb{Z}^2$ in the first quadrant of the plane. Then, traveling along the line $y = \alpha x + \beta$ starting with $x = 0$, write down the letter 1 each time a vertical grid line is crossed, and the letter 2 each time a horizontal grid line is crossed [**86**, Chapter 6]. If α is a quadratic irrational, sturmian sequences can often be related to substitutions [**170**]. For instance, if $\beta = 0$ and the slope $\alpha \in (0,1)$ of the line is a *Sturm number*, *i.e.*, a quadratic irrational whose Galois conjugate α' satisfies $\alpha' \notin (0,1)$, we know that the associated sturmian cutting sequence is the fixed point of a substitution. The most famous case is the fixed point of the Fibonacci substitution $\sigma(1) = 12$, $\sigma(2) = 1$ which is associated with $\alpha = \frac{1+\sqrt{5}}{2}$ [**20, 71**]. If the slope α is not a quadratic number, then a recoding process is used in order to describe the factors of the sequence by suitable compositions of two "basic" substitutions (see [**86**, Chapter 6]). The complexity properties of sturmian sequences are used in a variety of applications, such as compression to recover repetitions in DNA sequences [**73**] or optimal allocation in networks [**89**].

1.1.2. Number theory. – Since the time when they first appeared, substitutions have been deeply related to number theory: already Thue observed that the Thue-Morse sequence classifies integers with respect to the parity of the sum of their binary digits. The Baum-Sweet sequence describes whether the binary expansion of a positive integer contains at least one odd string of zeros. It is obtained as the projection of a substitution on a 4 letter alphabet [**21, 38**]. A bridge between substitutions and number theory is also given by the Cobham Theorem, which states that an infinite sequence $(u_k)_{k\geq 0}$ is a letter-to-letter projection of the fixed point of a substitution of constant length b if and only if the letter u_k of the sequence is produced by feeding a finite automaton with the expansion of k in base b [**64**]. This theorem allows to derive deep transcendence properties: for instance, the real numbers with continued fraction expansions given by the Thue-Morse sequence, the Baum-Sweet sequence, or the Rudin-Shapiro sequence (see *e.g.* [**21, 146**] for its definition) are all transcendental, the proof being based on the "substitutive" structure of these sequences [**2**]. Additionally, irrational numbers whose binary expansion is given by the fixed point of a substitution are all transcendental [**1**]. In the field of diophantine approximation, substitutions produce transcendental numbers which are very badly approximable by cubic algebraic integers [**145**]; the description of greedy expansions of reals in non-integer base [**6, 166**] by means of substitutions also results in best approximation