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QUANTUM GROUPS AND KNOT INVARIANTS

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Ce livre propose une introduction aux groupes quantiques, aux catégories monoïdales tressées et aux invariants quantiques de nœuds et de variétés de dimension trois. Nous mettons l'accent sur les relations profondes qui existent entre ces domaines et qui ont été découvertes au cours de la dernière décennie.

Dieses Buch bietet eine Einleitung in die Quantengruppen, die verzopften monoidalen Kategorien und die Quanteninvarianten von Knoten und von dreidimensionalen Mannigfaltigkeiten. Die tiefliegenden Beziehungen, die neuerdings zwischen diesen Bereichen entdeckt wurden, werden hier unterstrichen.

This book provides a concise introduction to quantum groups, braided monoidal categories, and quantum invariants of knots and of three-dimensional manifolds. The exposition emphasizes the newly discovered deep relationships between these areas.

В книге дается сжатое введение в теорию квантовых групп, косовых категорий и квантовых инвариантов узлов и трехмерных многообразий. Особое внимание уделяется недавно открытым глубоким взаимосвязям между этими областями.

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CONTENTS

INTRODUCTION

In this book we survey recent spectacular developments in the theory of Lie algebras and low-dimensional topology. These developments center around quantum groups and braided categories on the algebraic side and new invariants of knots, links, and three-manifolds on the topological side. This new theory has been, to a great extent, inspired by ideas that arose in theoretical physics. This strongly emphasizes a remarkable unity between the theory of Lie algebras, low-dimensional topology, and mathematical physics.

Quantum groups were introduced around 1983–1985 by V. Drinfeld and M. Jimbo. They may be roughly described as one-parameter deformations of the enveloping algebras of semisimple Lie algebras. Quantum groups appeared as an algebraic formulation of the work of physicists, especially from the Leningrad school of L. Faddeev, on the Yang-Baxter equation. An important feature of quantum groups is that the category of their representations has a so-called braiding. The notion of a braided monoidal category, formulated by A. Joyal and R. Street, plays a fundamental rôle in this theory.

An independent breakthrough was made in knot theory in 1984 by V. Jones. He used von Neumann algebras to define a new polynomial invariant of links. The study of the Jones polynomial rapidly involved ideas of statistical mechanics including the Yang-Baxter equation. It was observed by N. Reshetikhin and V. Turaev that the braided categories derived from quantum groups were the right algebraic objects needed to define representations of the braid groups and link invariants. This leads to a vast set of polynomial invariants of links whose components are coloured with finite-dimensional representations of a complex semisimple Lie algebra. This generalizes the Jones polynomial, which arises when all components of a link are coloured with the fundamental two-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$.

Further study proceeded in several different, albeit related, directions. In 1988 E. Witten invented the notion of a topological quantum field theory and outlined a fascinating picture of such a theory in three dimensions. This picture includes a path integral definition of numerical invariants of three-manifolds and links in three-manifolds generalizing the values of the Jones polynomial at the roots of unity. A rigourous mathematical definition of such three-manifold invariants was given by N. Reshetikhin and V. Turaev in 1988 on the basis of the theory of quantum groups at roots of unity.

At about the same time (1989-90), in the quite different context of singularity theory, V. Vassiliev introduced the notion of a knot invariant of finite degree. The invariants of degree $0, 1, 2, \ldots$ form an increasing filtration on the vector space of all knot invariants. The associated graded vector space can be described in terms of chord diagrams which may be viewed as a mathematical version of Feynman diagrams. In 1992 M. Kontsevich constructed a universal Vassiliev-type invariant of knots with values in the algebra of chord diagrams. This invariant dominates all finite degree invariants. The polynomial invariants of knots derived from quantum groups can also be computed from the Kontsevich invariant. Indeed, these polynomials may be expanded as formal series whose coefficients are invariants of finite degree.

This survey is intended to introduce the reader in the world of quantum groups, braided categories, knots, three-manifolds, and their invariants. We have not tried to give a complete picture of the theory, but rather to highlight its main features. Unfortunately, we had to leave out of the scope of this book a number of important aspects of the theory, including Woronowicz's approach to quantum groups in the framework of operator algebras, the dual construction by Faddeev, Reshetikhin, and Takhtadjian, the connections with state sum models of statistical mechanics, the quantization of Poisson structures and the theory of Poisson-Lie groups.

The book is organized as follows. We start in Chapter I with the Yang-Baxter equation. We show how solutions of this equation lead to representations of the braid groups. In Chapter II we introduce the concept of a braided bialgebra and show that the category of representations of such a bialgebra is a braided monoidal category. Examples of braided bialgebras are provided by the quantum groups defined and studied in Chapters III and IV.

In Chapter III we present Drinfeld's quantum double construction, which is a general method to produce braided bialgebras. Quantum groups appear in Chapter IV with the quantization $U_q\mathfrak{sl}(N+1)$ of the Lie algebra $\mathfrak{sl}(N+1)$. We give an overview of their main properties.

In Chapter V we enter the world of low-dimensional topology. We start with a definition of the Jones polynomial using the Kauffman bracket. We show that the study of knots, links, and more general objects called tangles naturally leads to braided monoidal categories. This yields geometric constructions of such categories.

In Chapter VI we introduce an important class of braided monoidal categories, namely the ribbon categories. Monoidal categories derived from the representations of quantum groups or from tangles are ribbon categories. We explain how to construct isotopy invariants of knots, links, and tangles, whose components are coloured with objects of a ribbon category. Then we introduce the more restricted class of modular categories and show how to derive from each modular category the corresponding Reshetikhin-Turaev invariant of three-manifolds and links in three-manifolds.

In Chapter VII we survey the representation theory of $U_q\mathfrak{sl}(N+1)$ and show that it leads to ribbon and modular categories, hence to the construction of "quantum invariants" of links and three-manifolds.

Chapter VIII is devoted to the theory of Vassiliev invariants of links. Examples of

Vassiliev invariants are provided by the quantum invariants of the previous chapter. We formulate an important theorem due to Kontsevich.

In Chapter IX we present more advanced topics based on work of Drinfeld. In particular, we give a construction of Kontsevich's universal link invariant and show how to recover the quantum invariants from it. In the very last section we explain how the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the space of Vassiliev invariants.

We close with a guide to the literature for the reader wishing to get more information on the subject.

This book grew out of notes we wrote for the "Journées Quantiques" that took place at the Department of Mathematics of the Université Louis Pasteur in Strasbourg on April 2–4, 1993. This was the first of a series of semi-annual meetings "État de la Recherche" initiated by the Société Mathématique de France (SMF) and sponsored by the SMF, the Ministère de la Recherche et de la Technologie, the Ministère de l'Éducation Nationale (DRED), and the Institut de Recherche Mathématique Avancée (Strasbourg).

Chapter 1

THE YANG-BAXTER EQUATION AND BRAID GROUP REPRESENTATIONS

In this chapter we introduce the Yang-Baxter equation and show how any solution of this equation gives rise to representations of the braid groups.

1. The Yang-Baxter Equation

***1.1. R-matrices.* —** Consider a vector space V over a field k. The *Yang-Baxter equation* is the following equation for a linear automorphism c of $V \otimes V$:

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c). \tag{1.1}$$

Equation (1.1) holds in the automorphism group of $V \otimes V \otimes V$. A solution is called an *R-matrix*. Let $(v_i)_i$ be a basis of the vector space V. An automorphism c of $V \otimes V$ is defined by the family $(c_{ij}^{k\ell})_{i,j,k,\ell}$ of scalars determined by

$$c(v_i \otimes v_j) = \sum_{k,\ell} c_{ij}^{k\ell}\, v_k \otimes v_\ell .$$

Then c is a solution of the Yang-Baxter equation if and only if, for all i, j, k, ℓ, m, n,

$$\sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{\ell m} = \sum_{y,q,r} c_{jk}^{qr} c_{iq}^{\ell y} c_{yr}^{mn} . \tag{1.2}$$

Solving the non-linear equations (1.2) is a highly non-trivial problem. Nevertheless, numerous solutions of the Yang-Baxter equation have been discovered in the 1980's. Let us list a few simple ones.

***1.2. Example.* —** For any vector space V we denote by $\tau_{V,V} \in \mathrm{Aut}(V \otimes V)$ the *flip* switching the two copies of V. It is defined by

$$\tau_{V,V}(v_1 \otimes v_2) = v_2 \otimes v_1 \quad \text{for any } v_1, v_2 \in V.$$

The flip satisfies the Yang-Baxter equation because of the Coxeter relation

$$(12)(23)(12) = (23)(12)(23)$$

holding in the symmetric group S_3, where (ij) denotes the transposition exchanging i and j.

1.3. Example. — Let V be a finite-dimensional vector space with a basis $(e_1, \ldots, e_N)$. For any invertible scalar q, we define an automorphism c of $V \otimes V$ by

$$c(e_i \otimes e_j) = \begin{cases} q\, e_i \otimes e_i & \text{if } i = j, \\ e_j \otimes e_i & \text{if } i < j, \\ e_j \otimes e_i + (q - q^{-1})\, e_i \otimes e_j & \text{if } i > j. \end{cases} \tag{1.3}$$

1.4. Proposition. — *The automorphism c is a solution of the Yang-Baxter equation. Moreover, we have*

$$(c - q\,\mathrm{id}_{V\otimes V})(c + q^{-1}\,\mathrm{id}_{V\otimes V}) = 0.$$

We leave the proof as an exercise.

Observe that Example 1.3 is a one-parameter-deformation of the automorphism $\tau_{V,V}$ of Example 1.2. To recover the latter, set $q = 1$ in (1.3). Note that, when $N = 2$, the matrix of the automorphism c in the basis formed by the vectors $(v_1 \otimes v_1, v_2 \otimes v_2, v_1 \otimes v_2, v_2 \otimes v_1)$ is

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & q - q^{-1} \end{pmatrix}. \tag{1.4}$$

1.5. Exercises.

(a) Resume the hypotheses and the notations of Example 1.3, which we generalize using two invertible scalars p, q and a family $\{r_{ij}\}_{1\leq i,j\leq N}$ of scalars such that $r_{ii} = q$ and $r_{ij}r_{ji} = p$ when $i \neq j$. Define an automorphism c of $V \otimes V$ by

$$c(e_i \otimes e_j) = \begin{cases} qe_i \otimes e_i & \text{if } i = j, \\ r_{ji}e_j \otimes e_i & \text{if } i < j, \\ r_{ji}e_j \otimes e_i + (q - pq^{-1})e_i \otimes e_j & \text{if } i > j. \end{cases}$$

Check that the automorphism c is an R-matrix such that

$$(c - q\,\mathrm{id}_{V\otimes V})(c + pq^{-1}\,\mathrm{id}_{V\otimes V}) = 0.$$

(b) Consider a matrix of the form

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

Prove that it is an R-matrix if and only if the following relations hold:

$$\begin{gathered} adb = adc = ad(a-d) = 0, \\ p^2a = pa^2 + abc, \quad q^2a = qa^2 + abc, \\ p^2d = pd^2 + dbc, \quad q^2d = qd^2 + dbc. \end{gathered}$$

2. Artin's Braid Groups

2.1. Definition. — Fix an integer $n \geq 3$. We define the *braid group* with n strands as the group B_n generated by $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ and the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j| > 1, \tag{2.1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } 1 \leq i,j \leq n-1. \tag{2.2}$$

When $n = 2$, we define B_2 as the free group on one generator σ_1. It is useful to set $B_0 = B_1 = \{1\}$.

There is a natural surjection of groups from B_n to the symmetric group S_n, that is the group of all permutations of the set $\{1, \ldots, n\}$. Indeed, consider the $(n-1)$ transpositions

$$s_i = (i, i+1) \qquad (i = 1, \ldots, n-1).$$

They clearly satisfy Relations (2.1–2.2). It follows that there exists a unique group morphism $\pi: B_n \to S_n$ such that $\pi(\sigma_i) = s_i$ for all i. This morphism is surjective because the transpositions s_i form a generating set for S_n. Actually, to pass from a presentation of the group B_n to a presentation of S_n, it suffices to add the relations

$$\sigma_i^2 = 1 \qquad (i = 1, \ldots, n-1).$$

One big difference between symmetric groups and braid groups is that the former are finite groups while the latter are infinite groups when $n > 1$. Moreover, the group B_n is torsion-free, that is to say, all elements $\neq 1$ have infinite order.

2.2. From the Yang-Baxter equation to representations of the braid groups. — Let V be a vector space and c an automorphism of $V \otimes V$ that is an R-matrix as defined in Section 1. For $1 \leq i \leq n-1$, define a linear automorphism c_i of the n-fold tensor power $V^{\otimes n}$ by

$$c_i = \begin{cases} c \otimes \mathrm{id}_{V^{\otimes(n-2)}} & \text{if } i = 1, \\ \mathrm{id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{id}_{V^{\otimes(n-i-1)}} & \text{if } 1 < i < n-1, \\ \mathrm{id}_{V^{\otimes(n-2)}} \otimes c & \text{if } i = n-1. \end{cases} \tag{2.3}$$

We claim that Relations (2.1–2.2) hold for the automorphisms $c_1, \ldots, c_{n-1}$. This is immediate for (2.1). As for (2.2), observe that it suffices to check that $c_1c_2c_1 = c_2c_1c_2$ is satisfied in $\mathrm{Aut}(V \otimes V \otimes V)$, but this is another way to write the Yang-Baxter equation.

This proves the following.

2.3. Proposition. — *Let $c \in \mathrm{Aut}(V \otimes V)$ be a solution of the Yang-Baxter equation. Then, for any $n > 0$, there exists a unique homomorphism $\rho_n^c : B_n \to \mathrm{Aut}(V^{\otimes n})$ such that $\rho_n^c(\sigma_i) = c_i$ for $i = 1, \ldots, n-1$.*

3. Alternative description of B_n

In Section 2 we gave an algebraic definition of the braid group. We now present Artin's original geometric definition and explain the term "braid".

Consider the product $\mathbb{C}^n$ of n copies of the complex line. Inside it we define Y_n as the subset of all n-tuples $(z_1, \ldots, z_n)$ of points of $\mathbb{C}$ such that $z_i \neq z_j$ when $i \neq j$. We distinguish a point $p = (1, \ldots, n)$ in Y_n. The symmetric group S_n acts on Y_n by permutations of the coordinates.

The quotient space $X_n = Y_n/S_n$ is the *configuration space of n points in $\mathbb{C}$*.

3.1. Theorem. — *The fundamental group $\pi_1(X_n, p)$ of the configuration space X_n is isomorphic to the braid group B_n.*

This theorem is due to E. Artin. We shall content ourselves with exhibiting a homomorphism from B_n to $\pi_1(X_n, p)$. To the generator σ_i of B_n we assign the continuous map $f = (f_1, \ldots, f_n) : [0,1] \to \mathbb{C}^n$ defined for $s \in [0,1]$ and all j by

$$f_i(s) = \tfrac{1}{2}\big(2i + 1 - \exp(\sqrt{-1}\,\pi s)\big),$$

$$f_{i+1}(s) = \tfrac{1}{2}\big(2i + 1 + \exp(\sqrt{-1}\,\pi s)\big),$$

$$f_j(s) = j \quad \text{if } j \neq i, i+1.$$

It is easy to check that f is a loop at the point p in the configuration space X_n. Let $\widehat{\sigma}_i$ be its class in $\pi_1(X_n, p)$. The elements $\widehat{\sigma}_1, \ldots, \widehat{\sigma}_{n-1}$ satisfy Relations (2.1) and (2.2). Thus, by definition of B_n, there exists a unique homomorphism $B_n \to \pi_1(X_n, p)$ sending σ_i to $\widehat{\sigma}_i$ for all $i = 1, \ldots, n-1$. This homomorphism is in fact an isomorphism. For details, see [Bir74] or [BZ85].

We now wish to give a more familiar image of the braid group. Let

$$f = (f_1, \ldots, f_n) : [0,1] \longrightarrow Y_n \subset \mathbb{C}^n$$

be a continuous map representing an element of $\pi_1(X_n, p)$, hence of B_n. Consider the subset L_f of $\mathbb{C} \times [0,1]$

$$L_f = \bigcup_{i=1}^{n} \big\{(f_i(s), s) \mid s \in [0,1]\big\}.$$

It is the disjoint union of n intervals continuously embedded in $\mathbb{C} \times [0,1]$. We call it a *braid with n strands*. Note that

(i) the boundary of L_f is the set $\{1, \ldots, n\} \times \{0,1\}$ and

(ii) for all $s \in [0,1]$ the intersection of L_f with $\mathbb{C} \times \{s\}$ consists of exactly n points.

Conversely, any disjoint union of n intervals continuously embedded in $\mathbb{C} \times [0,1]$ such that Conditions (i) and (ii) hold is a subset L_f for some loop f of X_n.

It follows that there is an equivalence relation on braids with n strands such that $B_n \cong \pi_1(X_n, p)$ is in bijection with the set of equivalence classes of braids with n strands. This equivalence is called *isotopy*. We shall encounter isotopy again in Section 1 of Chapter 5 when we introduce tangles, which generalize braids as well as knots.

***3.2 Exercise.* —** Describe the group structure induced by B_n on the set of braids with n strands.