

# *Astérisque*

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**ASTÉRISQUE**

**1992**

**AN EXTENSION OF A THEOREM  
BY CHEEGER AND MÜLLER**

**Jean-Michel BISMUT and Weiping ZHANG  
(with and appendix by François LAUDENBACH)**

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**This volume is dedicated to Jeff Cheeger and Werner Müller**



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# *Astérisque*

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WEIPING ZHANG

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