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1. Introduction: terminology and facts.– Let E be an elliptic curve defined over $\mathbb{Q}$ which is supposed to be modular, i.e. E is a Weil curve, and denote by $F(z) = \sum A_n e^{2\pi i n z}$ the weight 2 newform attached to E by the Eichler-Shimura congruences.

Fix a prime $p > 7$. We shall be interested in which cases E has additive reduction at p, excluding the Kodaira reduction types I_ν^* $(\nu \geq 0)$ which are related to the potentially semi-stable case. Thus p divides exactly twice the geometric conductor N_E of the elliptic curve E.

After [Ed 89] we say that E is p-vertical if E has bad but potentially good ordinary reduction at p, and that E is p-horizontal if E has bad but potentially good supersingular reduction at p. Recall that these conditions can be given in terms of the Hasse invariant (cf. [Hu 87], pag.248) of E and, moreover, one gets:

$$E \text{ is } p\text{-vertical} \iff p \equiv 1 \pmod{e},$$

where e is the least common multiple of the multiplicities of the irreducible components of the special fibre of the stable model; that is

$$e = \begin{cases} 6 & \text{if } p\text{-type}\,(E) = II, II^*\,; \\ 4 & \text{if } p\text{-type}\,(E) = III, III^*\,; \\ 3 & \text{if } p\text{-type}\,(E) = IV, IV^*\,. \end{cases}$$

The Galois module E_p of the p-torsion points of E gives rise to a continuous and odd representation

$$\rho :\ \mathrm{G}_{\mathbb{Q}} \rightarrow \mathrm{Aut}(E_p) \simeq \mathrm{GL}_2(\mathbb{F}_p),$$

which is almost always absolutely irreducible (cf. [Ma 78]).

Serre's conjecture (3.2.4?) in [Se 87] predicts, in this case, the existence of a Hecke cusp form $(\bmod\, p)$

$$f(q) = \sum a_n q^n$$

of type $(N_\rho, k_\rho, \varepsilon_\rho)$ satisfying

$$a_n \equiv A_n \pmod p\,, \qquad \text{for all } n \text{ prime to } N_E\,.$$

The level N_ρ, the weight k_ρ and the character ε_ρ are given by a precise recipe in [Se 87] and, depending on the Néron model of E, they have been computed, for instance in [Ba-La 91].

In [Ba-La 91], we verify (3.2.4?) for the Galois representations defined by the p-torsion points of the p-vertical Weil curves. In this paper our purpose is to emphasize the difference between the p-vertical and the p-horizontal cases in order to check Serre's conjecture. Several numerical examples, collected by computer calculations, lead us to give a conjecture which implies (3.2.4?) for the horizontal case.

2. Lowering the level (ordinary case).– First, we shall consider a general situation. Let

$$F(z) = \sum_{n=1}^{\infty} A_n e^{2\pi i n z}$$

be a newform of type (N, k, ε), defined over $\overline{\mathbb{Q}}$. If $\alpha : (\mathbb{Z}/M\mathbb{Z})^* \to \mathbb{C}^*$ is a Dirichlet character modulo M, then the twisted form

$$F \otimes \alpha\,(z) = \sum_{n=1}^{\infty} A_n\, \alpha(n) e^{2\pi i n z} \qquad (\alpha(n) = 0 \text{ if g.c.d.}\,(n, M) \neq 1)$$

is a Hecke cusp form of type $(N', k, \varepsilon\, \alpha^2)$, where

$$N' = \text{l.c.m.}\,(N, M \cdot \text{conductor}\,(\varepsilon), M^2)\,.$$

If M is prime to the level N, then $F \otimes \alpha$ is a newform of type $(NM^2, k, \varepsilon\,\alpha^2)$; otherwise, the form $F \otimes \alpha$ can either be or not be a newform! The question is to decide when it is.

After Li's work [Li 75], one has a nice criterion to decide whether a Hecke cusp form is new or not. More precisely, consider the operators K and H_N defined by

$$G|H_N = G|\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \quad \text{and} \quad G|K(z) = \overline{G(-\overline{z})}\,,$$

for each cusp form $G(z) = \sum_{n=1}^{\infty} B_n e^{2\pi i n z}$ of type (N, k, ε). We have

PROPOSITION. (cf. [Li 75]). *Let $G \in S_k(N, \varepsilon)$ be a Hecke cusp form. Then G is a newform of type (N, k, ε) if and only if the functional equation $G|K|H_N = \gamma G$ holds for a certain complex constant γ of absolute value 1.*

Let us go back to the case of elliptic curves. Let F be the newform attached to the Weil curve E as above and let $N_E = Np^2$ be the conductor of E. Choose an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ and let $\psi : (\mathbb{Z}/p\mathbb{Z})^* \to \overline{\mathbb{Q}}$ be the Dirichlet character which satisfies

$$\psi(n)\, n \equiv 1 \pmod{\mathfrak{P}},$$

where $\mathfrak{P}$ is the place of $\overline{\mathbb{Q}}$ dividing p fixed by our embedding.

We ask for which values of $j \in \{0, \dots, p-2\}$ the twisted form $F \otimes \psi^j$ fails to be new. Prof. D. B. Zagier suggested to us to apply the following test which is an immediate consequence of Li's result.

COROLLARY. *Keep the above notations. If there exist a complex number $z \in \mathbb{H}$ such that*

$$\left| \frac{\sum_{n=1}^{\infty} \overline{A}_n \overline{\psi}^j(n)\, e^{-2\pi i n / N_E z}}{N_E\, z^2 \sum_{n=1}^{\infty} A_n \psi^j(n)\, e^{2\pi i n z}} \right| - 1 \neq 0,$$

then $F \otimes \psi^j$ is not new.

On a VAX 8600 at the Facultat d'Informàtica de Barcelona we have obtained the following numerical examples, by taking $z = 2i/\sqrt{N_E} \in \mathbb{H}$ and a few number (around 500) of Fourier coefficients for $F \otimes \psi^j$.

For the elliptic curve 338 **A1** (cf. [Cre 91]),

$$E : y^2 + xy = x^3 - x^2 + x + 1$$

of conductor $N_E = 2 \cdot 13^2$, we get the following data:

j	TEST
0	0.0000000000
1	0.0000000000
2	2.6699959395
3	0.0000000000
4	0.0000000000
5	0.0000000000
6	0.0000000000
7	0.0000000000
8	0.0000000000
9	0.0000000000
10	2.6699959395
11	0.0000000000

Observe that the Hecke cusp forms $F \otimes \psi^2$ and $F \otimes \psi^{10}$ don't seem to be newforms. Indeed, as we shall see later, they are not newforms.

Another example is provided by the elliptic curve

$$E \,:\, y^2 + xy + y = x^3 - 39x - 27$$

of conductor $N_E = 43^2$ (cf. [Ed-Gr-To 90]). In this case we get:

j	TEST
14	5.5513673695
28	5.5513673695 ,

and zero for all the others values of j. Now, the Hecke cusp forms $F \otimes \psi^{14}$ and $F \otimes \psi^{28}$ are not newforms.

In the previous examples E is vertical at $p = 13$, 43, respectively; indeed, in both cases we have $p \equiv 1 \pmod 3$ and, following Tate's algorithm [Ta 75], we find that p-type (E) is equal to II and IV, respectively.

Actually, we are able to say what happens in the general p-vertical case. If ℓ denotes a prime which does not divide the conductor of E, then one can prove that the restriction to an inertia group I_p at p of the ℓ-adic representation ρ_ℓ attached to $F \otimes \psi^{\frac{p-1}{e}}$ is given by

$$\rho_\ell(I_p) = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} .$$

Therefore, we obtain

PROPOSITION. (cf. [Ba-La 91]). *If E is a p-vertical Weil curve as above, there exists a newform*

$$G(z) = \sum B_n e^{2\pi i n z} \in S_2(Np, \psi^{2\frac{p-1}{e}}),$$

having the same eigenvalue system as the twisted form $F \otimes \psi^{\frac{p-1}{e}}$.

Similar arguments do not work when E is p-horizontal. Consider the following example: the elliptic curve 605 **A1**, in [Cre 91],

$$E \,:\, y^2 + xy = x^3 - x^2 - 1414x - 44027$$

of conductor $N_E = 5 \cdot 11^2$ has 11-type IV^*. Since $11 \equiv 2 \pmod 3$, E is 11-horizontal. Running our program we get: