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Contents

Introduction

There are many different types of algebra: associative, associative and commutative, Lie, Poisson, etc., etc. Each comes with an appropriate notion of a module and thus with an associated theory of representations. Moreover, as is becoming more and more important in a variety of fields, including algebraic topology, algebraic geometry, differential geometry, and string theory, it is very often necessary to deal with "algebras up to homotopy" and with "partial algebras". The associated theories of modules have not yet been developed in the published literature, but these notions too are becoming increasingly important. We shall study various aspects of the theory of such generalized algebras and modules in this paper. We shall also develop some related algebra in the classical context of modules over DGA's. While much of our motivation comes from the theory of mixed Tate motives in algebraic geometry, there are pre-existing and potential applications in all of the other fields mentioned above.

The development of abstract frameworks in which to study such algebras has a long history. It now seems to be widely accepted that, for most purposes, the most convenient setting is that given by operads and their actions [46]. While the notion was first written up in a purely topological framework, due in large part to the resistance of topologists to abstract nonsense at that period, it was already understood by 1971 that the basic definitions apply equally well in any underlying symmetric monoidal (= tensor) category [35]. In fact, certain chain level concepts, the PROP's and PACT's of Adams and MacLane [42], were important precursors of operads. From a topological point of view, the switch from algebraic to topological PROP's, which was made by Boardman and Vogt [11], was a major step forwards. Perhaps for this reason, a chain level algebraic version of the definition of an operad did not appear in print until the 1987 paper of Hinich and Schechtman [31]. Applications of such algebraic operads and their actions have appeared in a variety of contexts in other recent papers, for example [27, 28, 29, 32, 34, 33, 59].

In the algebraic setting, an operad $\mathscr{C}$ consists of suitably related chain complexes $\mathscr{C}(j)$ with actions by the symmetric groups Σ_j. An action of $\mathscr{C}$ on a chain complex A is specified by suitably related Σ_j-equivariant chain maps

$$\mathscr{C}(j) \otimes A^j \to A,$$

where A^j is the j-fold tensor power of A. The $\mathscr{C}(j)$ are thought of as parameter complexes for j-ary operations. When the differentials on the $\mathscr{C}(j)$ are zero, we think of $\mathscr{C}$ as purely algebraic, and it then determines an appropriate class of (differential) algebras. When the differentials on the $\mathscr{C}(j)$ are non-zero, $\mathscr{C}$ determines a class of (differential) algebras "up to homotopy", where the homotopies are determined by the homological properties of the $\mathscr{C}(j)$. For example, we say that $\mathscr{C}$ is an E_∞ operad if each $\mathscr{C}(j)$ is Σ_j-free and acyclic, and we then say that A is an E_∞ algebra. An E_∞ algebra A has a product for each degree zero cycle of $\mathscr{C}(2)$. Each such product is unital, associative, and commutative up to all possible coherence homotopies, and all such products are homotopic. There is a long history in topology and category theory that makes precise what these "coherence homotopies" are. However, since the homotopies are all encoded in the operad action, there is no need to be explicit. There is a class of operads that is related to Lie algebras as E_∞ operads are related to commutative algebras, and there is a concomitant notion of a "strong homotopy Lie algebra". In fact, any type of algebra that is defined in terms of suitable identities admits an analogous "strong homotopy" generalization expressed in terms of actions by appropriate operads.

We shall give an exposition of the basic theory of operads and their algebras and modules in Part I. While we shall give many examples, the deeper aspects of the theory that are geared towards particular applications will be left to later Parts. In view of its importance to string theory and other areas of current interest, we shall illustrate ideas by describing the relationship between the little n-cubes operads of iterated loop space theory on the one hand and n-Lie algebras and n-braid algebras on the other. An operad $\mathscr{E}$ of topological spaces gives rise to an operad $C_\#(\mathscr{E})$ of chain complexes by passage to singular chains. On passage to homology with field coefficients, there results a purely algebraic operad $H_*(\mathscr{E})$. There is a particular operad of topological spaces, denoted $\mathscr{C}_n$, that acts naturally on n-fold loop spaces. For $n \geq 2$, the algebras defined by $H_*(\mathscr{C}_n; \mathbb{Q})$ are exactly the $(n-1)$-braid algebras. Even before doing any calculation, one sees from a purely homotopical theorem of [46] that, for any path connected space X, $H_*(\Omega^n\Sigma^n X; \mathbb{Q})$ is the free $H_*(\mathscr{C}_n; \mathbb{Q})$-algebra generated by $H_*(X; \mathbb{Q})$. This allows a topological proof, based on the Serre spectral sequence, of the algebraic fact that the free n-braid algebra generated by a graded vector space V is the free commutative algebra generated by the free n-Lie algebra generated by V. Actually, the results just summarized are the easy characteristic zero case of Cohen's much deeper calculations in arbitrary characteristic [15, 16], now over twenty years old.

Operads and their actions are specified in terms of maps that are defined on tensor products of chain complexes. In practice, one often encounters structures that behave much like algebras and modules, except that the relevant maps are only defined on suitable submodules of tensor products. For geometric intuition, think of intersection products that are only defined between elements that are in general position. Such partial algebras have been used in topology since the 1970's, for example in [48] and in unpublished work of Boardman and Segal. In Part II, we shall generalize the notions of algebras over operads and of modules over algebras over operads to the context of partially defined structures. Such partially defined structures are awkward to study algebraically, and it is important to know when they can be replaced by suitably equivalent globally defined structures. We shall show in favorable cases that partial algebras can be replaced by quasi-isomorphic genuine algebras over operads, and similarly for modules. When k is a field of characteristic zero, we shall show further that E_∞ algebras and modules can be replaced by quasi-isomorphic commutative algebras and modules and, similarly, that strong homotopy Lie algebras and modules can be replaced by quasi-isomorphic genuine Lie algebras and modules. The arguments work equally well for other kinds of algebras.

One of the main features of the definition of an operad is that an operad determines an associated monad that has precisely the same algebras. This interpretation is vital to the use of operads in topology. The proofs of the results of Part II are based on this feature. The key tool is the categorical "two-sided monadic bar construction" that was introduced in the same paper that first introduced operads [46]. This construction has also been used to prove topological analogs of many of the present algebraic results, along with various other results that are suggestive of further algebraic analogs [47, 49, 26, 52]. In particular, the proofs in Part II are exactly analogous to a topological comparison between Segal's Γ-spaces [58] and spaces with operad actions that is given in [26].

While these results can be expected to have other applications, the motivation came from algebraic geometry. For a variety X, Bloch [7] defined the Chow complex $\mathfrak{Z}(X)$. This is a simplicial abelian group whose homology groups are the Chow groups of X. It has a partially defined intersection product, and we show in Part II that it gives rise to a quasi-isomorphic E_∞ algebra, denoted $\mathscr{N}(X)$. After tensoring with the rationals, we obtain a commutative differential graded algebra (DGA) $\mathscr{N}_{\mathbb{Q}}(X)$ that is quasi-isomorphic to $\mathscr{N}(X) \otimes \mathbb{Q}$. The construction of these algebras answers questions of Deligne [20] that were the starting point of the present work. His motivation was