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NON-COMMUTATIVE VECTOR VALUED L_p-SPACES AND COMPLETELY p-SUMMING MAPS

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Abstract. — We introduce a non-commutative analog of Banach space valued L_p-spaces in the category of operator spaces. Thus, given a von Neumann algebra M equipped with a faithful normal semi-finite trace φ and an operator space E, we introduce the space $L_p(M, \varphi; E)$, which is an E-valued version of non-commutative L_p, and we prove the basic properties one should expect of such an extension (*e.g.* Fubini, duality, ...). There are two important restrictions for the theory to be satisfactory: first M should be injective, secondly E cannot be just a Banach space, it should be given with an operator space structure and all the stability properties (*e.g.* duality) should be formulated in the category of operator spaces.

This leads naturally to a theory of "completely p-summing maps" between operator spaces, analogous to the Grothendieck-Pietsch-Kwapień theory (*i.e.* "absolutely p-summing maps") for Banach spaces. As an application, we obtain a characterization of maps factoring through the operator space version of Hilbert space. More generally, we study the mappings between operator spaces which factor through a non-commutative L_p-space (or through an ultraproduct of them) using completely p-summing maps. In this setting, we also discuss the factorization through subspaces, or through quotients of subspaces of L_p-spaces.

Résumé **(Espaces L_p non-commutatifs à valeurs vectorielles et applications complètement p-sommantes)**. — Nous introduisons un analogue non-commutatif de la notion d'espace L_p à valeurs vectorielles dans la catégorie des espaces d'opérateurs. Plus précisément, étant donnés une algèbre de von Neumann M, munie d'une trace normale semie-finie et fidèle et un espace d'opérateurs E, nous introduisons l'espace $L_p(M, \varphi; E)$ qui est une version E-valuée d'espaces L_p non commutatif et nous prouvons les propriétés fondamentales que l'on est en droit d'attendre d'une telle extension (e.g. Fubini, dualité...). Il y a deux restrictions importantes pour que cette théorie tourne bien : d'abord M doit être injective, ensuite E ne peut pas être simplement un espace de Banach, il doit être muni d'une structure d'espace d'opérateurs et toutes les propriétés structurelles (e.g. la dualité) doivent être formulées dans la catégorie des espaces d'opérateurs.

Cela conduit naturellement à une théorie des applications « complètement p-sommantes » entre espaces d'opérateurs, analogue à la théorie de Grothendieck-Pietsch-Kwapień (*i.e.* les applications absolument p-sommantes) pour les Banach. Comme application, nous obtenons une caractérisation des applications qui se factorisent par la version « espace d'opérateurs » de l'espace de Hilbert (= l'espace OH). Plus généralement, nous étudions les applications entre espaces d'opérateurs qui se factorisent à travers un espace L_p-non commutatif (ou bien à travers un ultraproduit de tels espaces) dans le langage des applications complètement p-sommantes. Dans ce cadre, nous considérons aussi les factorisations (complètement bornées) à travers un sous-espace (ou un quotient de sous-espace) d'un espace L_p non commutatif.

Contents

INTRODUCTION

In standard Lebesgue integration, for any measure space (Ω, μ) and any Banach space E, we know how to define the Banach space $L_p(\Omega, \mu; E)$ of E-valued L_p-functions (for $1 \leq p \leq \infty$) using a well known construction attributed to Bochner. When $\Omega = \mathbb{N}$ (resp. $\Omega = \{1, \ldots, n\}$) equipped with the counting measure $\mu = \sum\limits_{k \in \Omega} \delta_k$, then $L_p(\Omega, \mu; E)$ is simply the space $\ell_p(E)$ (resp. $\ell_p^n(E)$) formed of all the sequences (x_k) with $x_k \in E$ such that $\sum \|x_k\|_E^p < \infty$, equipped with the norm

$$\|(x_k)\|_{\ell_p(E)} = \left(\sum \|x_k\|_E^p\right)^{1/p} \quad \left(\text{resp. } \|(x_k)\|_{\ell_p^n(E)} = \left(\sum_1^n \|x_k\|_E^p\right)^{1/p}\right).$$

The case of any discrete measure space is analogous.

The non-commutative analog of ℓ_p is the Schatten class S_p which is defined for $1 \leq p < \infty$ as the space of all compact operators T on ℓ_2 such that $\operatorname{tr}|T|^p < \infty$ and is equipped with the norm

$$\|T\|_{S_p} = (\operatorname{tr}|T|^p)^{1/p}$$

with which it is a Banach space. We will often denote this simply by $\|T\|_p$. For $p = \infty$, we denote by S_∞ the space of all compact operators on ℓ_2 equipped with the operator norm.

If H is any Hilbert space (resp. if $H = \ell_2^n$) we will denote by $S_p(H)$ (resp. S_p^n) the space of all operators $T\colon\ H \to H$ such that $\operatorname{tr}|T|^p < \infty$ and we equip it with the norm $(\operatorname{tr}|T|^p)^{1/p}$. If $p = \infty$, $S_\infty(H)$ (resp. S_∞^n) is the space of all compact operators on H, equipped with the operator norm.

More generally, given a von Neumann algebra M equipped with a faithful normal semi-finite trace φ, one can define a non-commutative version of L_p which we denote by $L_p(M, \varphi)$. When φ is finite, $L_p(M, \varphi)$ can be described simply as the completion of M equipped with the norm $x \to \varphi(|x|^p)^{1/p}$. In the special case $M = B(\ell_2)$ equipped with its classical (infinite but semi-finite) trace $x \to \operatorname{tr}(x)$, $L_p(M, \varphi)$ can be identified with S_p.

There is an extensive literature about these spaces, following the pioneering work of Segal, Dixmier, Kunze and Stinespring in the fifties (**[S]**, **[Di]**, **[Ku]**, **[St]**). (See *e.g.* **[N]**, **[FaK]**, **[H2]**, **[Ko]**, **[Te1]**-**[Te2]**, **[Hi]**).

Consider in particular the so-called hyperfinite factor R. This is the infinite tensor product of M_2 ($= 2 \times 2$ matrices) equipped with its normalized trace. This object is the non-commutative analog of the probability space $\Omega = \{-1, +1\}^{\mathbb{N}}$ equipped with its usual probability P (P is the infinite product of $(1/2)\delta_1 + (1/2)\delta_{-1}$). When $M = R$, the space $L_p(M, \varphi)$ appears as the non-commutative analog of $L_p(\Omega, P)$, or equivalently of $L_p([0,1], dt)$. In non-commutative integration theory, there seems to be no analog (as far as we know) of vector valued integration, and while S_p and $L_p(M, \varphi)$ appear as the "right" non-commutative counterpart to ℓ_p and $L_p([0,1], dt)$, there is a priori no analog for $\ell_p(E)$ and $L_p([0,1], dt; E)$ when E is a Banach space. The main goal of the present volume is to fill this gap. We will show that if M is hyperfinite (=injective by **[Co]**) and if E is an operator space, *i.e.* E is given as a closed subspace of $B(H)$ (for some Hilbert space H), then using complex interpolation (see below for more on this), we can define in a very natural way the space $L_p(M, \varphi; E)$ for $1 \le p < \infty$. When $(M, \varphi) = (B(\ell_2), \mathrm{tr})$, we obtain the space $S_p[E]$ which is a non-commutative analog of $\ell_p(E)$. Our theory of these spaces has all the properties one should expect, such as duality, Fubini's theorem, injectivity and projectivity with respect to E, and so on...But the crucial point is that we must always work with operator spaces and not only Banach spaces. The theory of operator spaces emerged rather recently (with its specific duality) in the works of Effros-Ruan **[ER1]**-**[ER7]** and Blecher-Paulsen **[BP]**, **[B1]**-**[B3]**. In this theory, bounded linear maps are replaced by *completely bounded* ones, isomorphisms by *complete isomorphisms* and isometric maps by *completely isometric ones.* In particular, given an operator space E, the spaces $S_p[E]$ and $L_p(M, \varphi; E)$ will be constructed not only as Banach spaces but as operator spaces. Moreover, all identifications will have to be "completely isometric" (as defined below) rather than just isometric.

For instance, the classical (isometric) duality theorem

$$\ell_p(E)^* = \ell_{p'}(E^*)$$

becomes in our theory the completely isometric identity

$$S_p[E]^* = S_{p'}[E^*]$$

where on both sides the dual is meant in the operator space sense: when E is an operator space, the dual Banach space E^* can be realized in a specific manner as a closed subspace of some $B(H)$, this is what we call the dual "in the operator space sense" (called the standard dual in **[BP]**); see below for background on this.

In a different direction, let (N, ψ) be another hyperfinite von Neumann algebra equipped with a faithful normal semi-finite trace. We will obtain completely isometric identities

$$L_p(M, \varphi; L_p(N, \psi)) = L_p(M \otimes N, \varphi \times \psi) = L_p(N, \psi; L_p(M, \varphi)).$$

Actually, the first one holds even if N is not assumed hyperfinite, see (3.6) and (3.6)'.

In addition, the resulting functor $E \to L_p(M, \varphi; E)$ is both injective and projective. By this we mean that if $F \subset E$ is a closed subspace (=operator subspace) then the inclusion $L_p(M, \varphi; F) \subset L_p(M, \varphi; E)$ is completely isometric and we have a completely isometric identification

$$L_p(M, \varphi; E/F) = L_p(M, \varphi; E)/L_p(M, \varphi; F).$$

To some extent our theory works in the non-hyperfinite case (see the discussion in chapter 3) but then the preceding injectivity (resp. projectivity) no longer holds if $p = 1$ (resp. $p = \infty$).

In the case $p = 1$ our results are essentially contained in the works of Effros-Ruan **[ER2, ER8]** on the operator space version of the projective tensor product, see also **[BP]**. Indeed, these authors introduced the operator space version of the projective tensor product $E \otimes^\wedge F$ of two operator spaces E, F. Then if X is a non-commutative L_1-space, the E-valued version of X can be defined simply as $X \otimes^\wedge E$. (Warning: In general this is not the Grothendieck projective product of X and E, but its analog in the category of operator spaces.) The case $p = \infty$ is also known: if E is finite dimensional (for simplicity) and if M is any von Neumann algebra, then the minimal tensor product $M \otimes_{\min} E$ is the natural non-commutative analog of $L_\infty(\Omega, \mu; E)$. What we do in this volume is simply to use the complex interpolation method (an approach that has already proved very efficient in the study of non-commutative L_p-spaces, *cf.* **[Ko]**, **[Te1]**) to define the non-commutative "E-valued" L_p-spaces for the intermediate values, i.e. for $1 < p < \infty$.

The first part of this volume (chapters 1 to 4) is devoted to the theory of the spaces $L_p(M, \varphi; E)$. We first concentrate on the discrete case in chapter 1, then in chapter 2, we describe the operator space structure of the usual (=commutative) L_p-spaces and its relation to the discrete non-commutative case. We consider the general case in chapter 3 and the duality in chapter 4.

The second part (chapters 5 to 7) is devoted mainly to "completely p-summing maps". These are a natural extension in our new setting of the "absolutely p-summing maps" studied by Pietsch and Kwapień (**[Pi]**, **[Kw1]**-**[Kw2]**), following Grothendieck's fundamental work on Banach space tensor products **[G]**.

In the third and final part (chapter 8), we try to illuminate our new theory in the light of numerous concrete examples linked with analysis. The main emphasis there is on Khintchine's inequalities for the Rademacher functions (which we denote by (ε_n)), and numerous variants of them involving Gaussian random variables or their analog in Voiculescu's "free" probability theory. If we identify (ε_n) with the sequence of coordinate functions on Ω, the classical Khintchine inequalities provide a remarkable isomorphic embedding

$$\ell_2 \subset L_p(\Omega, P),$$

taking the canonical basis of ℓ_2 to (ε_n) (here $0 < p < \infty$). This is very often used in analysis through the resulting isomorphic embedding

$$L_p([0,1]; \ell_2) \subset L_p([0,1] \times \Omega, dt \times dP).$$