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ON THE STRUCTURE OF SUM-FREE SETS, 2

by

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Abstract. — A finite set of positive integers is called sum-free if $\mathbb{A} \cap (\mathbb{A} + \mathbb{A})$ is empty, where $\mathbb{A} + \mathbb{A}$ denotes the set of sums of pairs of non necessarily distinct elements from $\mathbb{A}$. Improving upon a previous result by G.A. Freiman, a precise description of the structure of sum-free sets included in $[1, M]$ with cardinality larger than $0.4M - x$ for $M \geq M_0(x)$ (where x is an arbitrary given number) is given.

1. Introduction

A finite set of positive integers $\mathcal{A}$ is called **sum-free** if $\mathcal{A} \cap (\mathcal{A} + \mathcal{A})$ is empty, where $\mathcal{A} + \mathcal{A}$ denotes the set of sums of pairs of elements from $\mathcal{A}$.

Such sum-free sets have been considered by Cameron and Erdős (cf. [1]), and the first result concerning their structure has been obtained by Freiman (cf. [3]). It is clear that for odd n, the sets $\{1, 3, 5, \ldots, n\}$ and $\{\frac{n+1}{2}, \frac{n+3}{2}, \ldots, n\}$ are sum-free. Freiman showed that when $\mathcal{A}$ is included in $[1, n]$ and its cardinality is at least $5n/12 + 2$, then $\mathcal{A}$ is essentially a subset of the ones we just described. In an unpublished paper, Deshouillers, Freiman and Sós showed the following improvement.

Theorem 1.1. — *Let $\mathcal{A}$ be a sum-free set with minimal element m and maximal element M. Under the assumption that $A = \operatorname{Card} \mathcal{A} > 0.4M + 0.8$, we have either*

(i) : *all the elements of $\mathcal{A}$ are odd,*

(ii) : *the minimal element of $\mathcal{A}$ is at least A, and we have*

$$\operatorname{Card}(\mathcal{A} \cap [1, M/2]) \leq (M - 2A + 3)/4.$$

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Examples have been produced to show that all the bounds in the theorem are sharp. We are not going to discuss the bound in (ii), but show what may happen if the condition on A is relaxed: let s be a positive integer, and consider

$$\mathcal{A}_1 = \{s, s+1, \dots, 2s-1\} \cap \{4s-1, \dots, 5s-2\},$$

as well as

$$\mathcal{A}_2 = \{2, 3, 7, 8, 12, 13, \dots, 5k-3, 5k-2, \dots, 5s-3, 5s-2\}$$

it is easy to see that $\mathcal{A}_1$ and $\mathcal{A}_2$ are sum-free, that their cardinality, $2s$, is precisely equal to $0.4(5s-2)+0.8$, and that they are very far from satisfying properties (i) or (ii) from Theorem 1.1. A further example, with $A = 0.4M + 0.4$ is

$$\mathcal{A}_3 = \{1, 4, 6, 9, \dots, 5k-4, 5k-1, \dots, 5s-4, 5s-1\}$$

Our aim is to show that when A is not much less than $0.4M$, then the structure of a sum-free set is described by Theorem 1.1, or close to one of the previous examples. More precisely, we have the following

Theorem 1.2. — *Let x be a positive real numbers; there exist real number $M_0(x)$ and $C(x)$ such that for every sum-free set $\mathcal{A}$ with largest element $M \geq M_0(x)$ and cardinality $A \geq 0.4M - x$, at least one of the following properties holds true*

(i) : *all the elements of $\mathcal{A}$ are odd,*
(ii) : *all the elements of $\mathcal{A}$ are congruent to 1 or 4 modulo 5,*
(iii) : *all the elements of $\mathcal{A}$ are congruent to 2 or 3 modulo 5,*
(iv) : *the smallest elements of $\mathcal{A}$ is at least equal to A and we have $|\mathcal{A} \cap [1, M/2]| \leq (M - 2A + 3)/4$*
(v) : *$\mathcal{A}$ is included in $[\frac{M}{5} - C(x), \frac{2M}{5} + C(x)] \cup [\frac{4M}{5} - C(x), M]$.*

The constants $C(x)$ and $M_0(x)$ may be computed explicitly from our proof. However, they are not good enough to lead us to the structure of $\mathcal{A}$ when A is about $0.375M$, where new structures appear.

We may reduce the proof of Theorem 1.2 to the case when $\mathcal{A}$ contains at least one even element. From now on, we take this assumption for granted. The proof will be conducted according to the location of the smallest element m of $\mathcal{A}$: section 4 and 5 are devoted to show that m is around 1 or $M/5$, or that it is at least equal to A; the structure of $\mathcal{A}$ will be deduced from this location in section 6 and 7. Section 3 aims at filling the gap between the content of [3] and a proof of Theorem 1.1, as well as presenting in a simple frame some of the ideas that will be developed later on. In the next section, we present our notation as well as general results.

2. Notation - General results

Letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ with or without indices or other diacritical symbols denote finite sets of integers. Their cardinality is represented by $|\mathcal{A}|, |\mathcal{B}|, |\mathcal{C}|, \dots$ or $A, B, C, \dots$ with the same diacritical symbols. For a non empty set $\mathcal{B}$, we further let

$M(\mathcal{B})$: be its maximal element,
$m(\mathcal{B})$: be its minimal element,

$l(\mathcal{B})$: be its length, i.e. $M(\mathcal{B}) - m(\mathcal{B}) + 1$,
$d(\mathcal{B})$: be the gcd of all the differences $(b_i - b_j)$ between pairs of elements of $\mathcal{B}$,
$\mathcal{B}_+ : = \mathcal{B} \cap [1, +\infty[$.

The letter $\mathcal{A}$ is restricted to denote a non empty sum-free set of positive integers, and we let

$$\mathcal{A}_0 = \mathcal{A} \cap 2\mathbb{Z},\ \mathcal{A}_1 = \mathcal{A} \cap (2\mathbb{Z}+1),$$

$$\mathcal{A}^- = \mathcal{A} \cap [1, M/2],\ \mathcal{A}^+ = \mathcal{A} \cap [M/2, M],$$

M (resp. m, resp. $M_0, \dots$) denote $M(\mathcal{A})$ (resp. $m(\mathcal{A})$, resp. $M(\mathcal{A}_0) \cdots$).

By x we denote a real number larger than -1. All the constants $C_1, C_2, \dots$ depend on x at most, and their value may change from one section to the other. Further, when we say that a property holds for M sufficiently large, we understand that there exist $M_0(x)$ depending on x at most, such that the considered property holds for M at least equal to $M_0(x)$.

We turn now our attention towards general results that will be used systematically, beginning with section 4.

Definition 2.1. — *A set $\mathcal{A}$ of positive integers is said to satisfy the general assumptions if it is a sum-free set that contains at least one even element and has cardinality $A = 0.4M - x$.*

Proposition 2.1. — *If $\mathcal{A}$ satisfies the general assumptions and M is large enough, we have the following properties*

(i) : *$\mathcal{A}$ contains an odd number,*
(ii) : $d(\mathcal{A}) = 1$,
(iii) : *$\mathcal{A} \cap (\mathcal{A} - \mathcal{A})$ is empty,*
(iv) : $M - m \geq 2A - 2 \Longrightarrow |(\mathcal{A} - \mathcal{A})_+| \geq \frac{3}{2}A - 2$,
(v) : $M - m \leq 2A - 3 \Longrightarrow |(\mathcal{A} - \mathcal{A})_+| \geq (M - m + A - 1)/2$,
(vi) : *for any integers u and v : $|\mathcal{A} \cap [u, u+v]| \leq (v+m)/2$,*
(vii) : *for any integer u : $|\mathcal{A} \cap [u, u+2m]| \leq m$*

Proof

(i) If $\mathcal{A}$ contains only even numbers, then the set $\mathcal{A}/2 = \{a/2 | a \in \mathcal{A}\}$ is a sum-free set that is contained in $[1, M/2]$, and so its cardinality is at most $M/4 + 1$ as can be directly seen (cf. also [4]). But $|\mathcal{A}/2| = |\mathcal{A}| = 0.4M - x$ which is larger than $M/4+1$ when M is large enough.

(ii) The number $d(\mathcal{A})$ is defined in such a way that $\mathcal{A}$ is included in an arithmetic progression modulo $d(\mathcal{A})$. Since $\mathcal{A}$ contains an even number (by our general assumption) as well as an odd number (by (i)), we have $d(\mathcal{A}) \neq 2$. On the other hand, we cannot have $d(\mathcal{A}) \geq 3$, otherwise $\mathcal{A}$ would have at most $M/3 + 1$ elements, which would contradict our general assumptions. Thus, $d(\mathcal{A}) = 1$.

(iii) Let $b \in \mathcal{A} \cap (\mathcal{A} - \mathcal{A})$. We can find a_1, a_2, a_3 in $\mathcal{A}$ such that $b = a_1 = a_2 - a_3$. This implies $a_2 = a_1 + a_3$, which is impossible. Thus $\mathcal{A} \cap (\mathcal{A} - \mathcal{A})$ is empty, and our argument shows even that last condition implies that $\mathcal{A}$ is sum-free.

(iv) and (v) are straightforward application of the following result ([2] and [5]):

Lemma 2.1. — *Let $\mathcal{B}$ and $\mathcal{C}$ be to finite sets of integers with $m(\mathcal{B}) = m(\mathcal{C}) = 0$, and let $M(\mathcal{B},\mathcal{C})$ be $max(M(\mathcal{B}), M(\mathcal{C}))$.*
If $M(\mathcal{B},\mathcal{C}) \leq |\mathcal{B}| + |\mathcal{C}| - 3$, then we have $|\mathcal{B}+\mathcal{C}| \geq M(\mathcal{B}) + |\mathcal{C}|$.
If $M(\mathcal{B},\mathcal{C}) \geq |\mathcal{B}| + |\mathcal{C}| - 2$ and $d(\mathcal{B} \cup \mathcal{C}) = 1$, then we have $|\mathcal{B}+\mathcal{C}| \geq M(\mathcal{B}) + |\mathcal{C}| - 3 + min(|\mathcal{B}|, |\mathcal{C}|)$.

(vi) The result is obvious when $v \leq m$, so we way assume $v > m$. We let $\mathcal{B} = \mathcal{A} \cap [u, u+v-m]$ and $\mathcal{C} = \mathcal{A} \cap [u+m, u+v]$. Since $\mathcal{A}$ is sum-free and m is in $\mathcal{A}$, we have $|\mathcal{B}| + |\mathcal{C}| \leq v - m$. Combined with the trivial upper bound $|\mathcal{A}| \leq |\mathcal{B}| + m$ and $|\mathcal{A}| \leq |\mathcal{C}| + m$, this inequality leads us to (vi).
(vii) We apply the same argument as above, leading to $|\mathcal{B}| + |\mathcal{C}| \leq v - m = m$, and further notice that $\mathcal{A} \cap]u, u+2m]$ is the union of $\mathcal{B}$ and $\mathcal{C}$.

The next results are fairly simple.

Lemma 2.2. — *Let $\mathcal{B}$ be a finite set of integers such that $2|\mathcal{B}| > l(\mathcal{B})$. Then $\mathcal{B} - \mathcal{B}$ contains $[1, 2|\mathcal{B}| - l(\mathcal{B}) - 1]$.*

Proof. — We consider a positive integer y which is not the difference of two elements of $\mathcal{B}$. We way assume $\mathcal{B} \subset [1, l(\mathcal{B})]$ and let

$$\begin{aligned} \mathcal{B}_1 &= \mathcal{B} \cap [1, y] &&, \mathcal{B}_2 = \mathcal{B} \cap [y+1, l(\mathcal{B})], \\ \mathcal{B}_3 &= \mathcal{B} \cap [1, l(\mathcal{B}) - y] &&, \mathcal{B}_4 = \mathcal{B} \cap [l(\mathcal{B}) - y + 1, l(\mathcal{B})]. \end{aligned}$$

Since y is not difference of two elements of $\mathcal{B}$, the sets $\mathcal{B}_2$ and $\mathcal{B}_3 + y$ are disjoint so that we have

$$|\mathcal{B}_2| + |\mathcal{B}_3| \leq l(\mathcal{B}) - y.$$

This easily leads to

$$2|\mathcal{B}| = |\mathcal{B}_1| + |\mathcal{B}_2| + |\mathcal{B}_3| + |\mathcal{B}_4| \leq y + l(\mathcal{B}) - y + y = l(\mathcal{B}) + y,$$

whence the inequality $y \geq 2|\mathcal{B}| - l(\mathcal{B})$.

Lemma 2.3. — *Let $\mathcal{B} = \{b_1 < b_2 < \cdots < b_B\}$ and $\mathcal{D} = \{d_1 < \cdots < d_D\}$ be to sets of integers such that we have $b_{i+1} - b_i < l(\mathcal{D})$ for $1 \leq i \leq B-1$, and $card\mathcal{D} \geq l(\mathcal{D}) - C$. We have $|\mathcal{B} + \mathcal{D}| \geq (l(\mathcal{B}) + l(\mathcal{D}) + 1)(1 - 3C/l(\mathcal{D}))$*

Proof. — Let $l(\mathcal{D}) = d_D - d_1 + 1$. We show that for any integer $u \in [b_1 + d_1, b_B + d_1[$, the interval $[u, u + l(\mathcal{D})]$ contains at most $2C$ integers which are not in $\mathcal{B} + \mathcal{D}$. We define the integer i such that $b_i + d_1 \leq u < b_{i+1} + d_1$. Since $b_{i+1} - b_i$ is less than $l(\mathcal{D})$, the interval $[u, u + l(\mathcal{D})]$ is included in $[b_i + d_1, b_{i+1} + d_D]$, which contains only elements in $\{b_i, b_{i+1}\} + \mathcal{D}$, with at most $2C$ exception. Since $[b_1 + d_1, b_B + d_D]$ can be covered with at most $(b_B + d_D + b_1 + d_1 + 1)/l(\mathcal{D}) + 1$ intervals of length $l(\mathcal{D})$, we have

$$\begin{aligned} |\mathcal{B} + \mathcal{D}| &\geq l(\mathcal{B}) + l(\mathcal{D}) + 1 - ((l(\mathcal{B}) + l(\mathcal{D}) + 1)/l(\mathcal{D}) + 1)2C \\ &\geq (l(\mathcal{B}) + l(\mathcal{D}) + 1)(1 - 3C/l(\mathcal{D})). \end{aligned}$$